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The Einstein-Maxwell-Particle System in the York Canonical Basis of ADM Tetrad Gravity: II) The Weak Field Approximation in the 3-Orthogonal Gauges and Hamiltonian Post-Minkowskian Gravity: the N-Body Problem and Gravitational Waves with Asymptotic Background.

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Abstract

In this second paper we define a Post-Minkowskian (PM) weak field approximation leading to a linearization of the Hamilton equations of ADM tetrad gravity in the York canonical basis in a family of non-harmonic 3-orthogonal Schwinger time gauges. The York time ${}^3K$ (the relativistic inertial gauge variable, not existing in Newtonian gravity, parametrizing the family and connected to the freedom in clock synchronization, i.e. to the definition of the instantaneous 3-spaces) is put equal to an arbitrary numerical function. The matter are point particles, with a Grassmann regularization of self-energies, and the electro-magnetic field in the radiation gauge: a ultraviolet cutoff allows a consistent linearization, which is shown to be the lowest order of a Hamiltonian Post-Minkowskian (HPM) expansion.

We solve the constraints and the Hamilton equations for the tidal variables and we find Post-Minkowskian gravitational waves with asymptotic background (and the correct quadrupole emission formula) propagating on dynamically determined non-Euclidean 3-spaces. The conserved ADM energy and the Grassmann regularizzation of self-energies imply the correct energy balance. A generalized transverse-traceless gauge can be identified and the main tools for the detection of gravitational waves are reproduced in these non-harmonic gauges. In conclusion we get a PM solution for the gravitational field and we identify a class of PM Einstein space-times, which will be studied in more detail in a third paper together with the PM equations of motion for the particles and their Post-Newtonian expansion (but in absence of the electro-magnetic field).

Finally we make a discussion on the *gauge problem in general relativity* to understand which type of experimental observations may lead to a preferred choice for the inertial gauge variable ${}^3K$ in the PM space-times. In the third paper we will show that this choice is connected with the problem of dark matter.

I. INTRODUCTION

In this paper we will study the linearization of the Hamiltonian formulation of ADM tetrad gravity given in Ref.[1] (quoted as paper I in what follows) to get a Post-Minkowskian (PM) description of gravitational waves (GW) in non-harmonic gauges in asymptotically Minkowskian space-times with the asymptotic Minkowski metric used as a background. We are able to reproduce all the main properties of GW's, which are usually derived in the standard Lagrangian approach to GW's in harmonic gauges with a Post-Newtonian (PN) expansion (see Appendix A for a review).

We define a new Hamiltonian Post-Minkowskiam (HPM) approach to the description of GW's in which we do not assume the decomposition ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}$, but we use an asymptotic Minkowski background 4-metric ${}^4\eta_{\mu\nu(asym)}$ at spatial infinity in a certain family of asymptotically Minkowskian space-times. As shown in paper I, this approach is based on ADM tetrad gravity in these space-times followed by a canonical transformation to a York canonical basis, which diagonalizes the York-Lichnerowitz formulation of general relativity [2, 3] allowing a clean separation between physical tidal degrees of freedom and inertial gauge variables inside the gravitational field in a 3-covariant way inside the non-Euclidean instantaneous 3-spaces (the Cauchy surfaces for the tidal variables): while in special relativity its shape as a sub-manifold of Minkowski space-time is arbitrary [4] (the gauge freedom in the choice of the convention for clock synchronization), in general relativity the 3-space is dynamically determined [5] except for the trace of its extrinsic curvature, the inertial gauge variable called York time (it describes the general relativistic remnant of the gauge freedom in clock synchronization).

In paper I we have explicitly evaluated the Hamilton equations of ADM tetrad gravity coupled to dynamical matter in the Schwinger time gauges in the York canonical basis defined in Ref.[6]. This paper was the final development of previous researches [7–10] in canonical gravity. The matter consists of the electro-magnetic field in the radiation gauge and of N dynamical (not test) massive point particles [1]. The Grassmann-valued electric charges and signs of the energy regularize both the electro-magnetic and gravitational self-energies (see Refs.[4, 12–15] for the electro-magnetic case in special relativity) in the equations of motion for the N-body problem avoiding both the gravitational and electro-magnetic self-energies. As a consequence the action principle for the particles given in paper I is well posed (there are no essential singularities on the particle world-lines).

Dirac theory of constraints is taken into account at every step, in particular when making gauge-fixings (instead in the standard ADM approach and numerical gravity the gauge fixings are chosen only on the basis of convenience). This will allow to make a clear distinction between the instantaneous action-at-a-distance effects implied by the super-Hamiltonian and super-momentum constraints (like it happens in every gauge theory: with the electro-magnetic field in the radiation gauge this is implied by the Gauss law constraint in presence

[1] In a future paper we will try to describe compact extended objects and their self-gravity (see footnote 16 of Appendix A) starting from balls of perfect fluids, whose Lagrangian description will be the extension to tetrad gravity of the special relativistic one given in Ref[11]. The Grassmann regularization is a semi-classical way out from causality problems like the ones arising when considering the point limit of extended models for the electron in classical electro-dynamics.

of matter) and retarded tidal effects.

We use a family of non-harmonic 3-orthogonal Schwinger time-gauges in which the gauge fixings imply only elliptic-type equations at a given time for the gauge variables. As a consequence, we avoid the wave equations for gauge variables like the lapse and shift functions present in the harmonic gauges and requiring initial data in the asymptotic past.

ADM tetrad gravity is formulated in an arbitrary admissible 3+1 splitting of the globally hyperbolic space-time, i.e. in a foliation with instantaneous space-like 3-spaces tending to a Minkowski space-like hyper-plane at spatial infinity in a direction independent way: they correspond to a clock synchronization convention and each one of them can be used as a Cauchy surface for field equations. As shown in Refs.[1, 7] the absence of super-translations implies that the SPI group of asymptotic symmetries is reduced to the asymptotic ADM Poincare' group [2] and the allowed 3+1 splittings must have the instantaneous 3-spaces tending to asymptotic special-relativistic Wigner hyper-planes orthogonal to the ADM 4-momentum in a direction-independent way (see Ref.[4] for these non-inertial rest frames in special relativity). At spatial infinity there are asymptotic inertial observers, carrying a flat tetrad ϵ^μ_A (${}^4\eta_{\mu\nu}\, \epsilon^\mu_A\, \epsilon^\nu_B = {}^4\eta_{AB}$, $\epsilon^\mu{}_{\alpha\beta\gamma}\, \epsilon^\alpha_1\, \epsilon^\beta_2\, \epsilon^\gamma_3 = \epsilon^\mu_\tau$), whose spatial axes can be identified with the fixed stars of star catalogues.

We use radar 4-coordinates $\sigma^A = (\sigma^\tau = \tau; \sigma^r)$, $A = \tau, r$, adapted to the admissible 3+1 splitting of the space-time and centered on an arbitrary time-like observer $x^\mu(\tau)$ (origin of the 3-coordinates σ^r): they define a non-inertial frame centered on the observer, so that they are *observer and frame- dependent*. The time variable τ is an arbitrary monotonically increasing function of the proper time given by the atomic clock carried by the observer. The instantaneous 3-spaces identified by this convention for clock synchronization are denoted Σ_τ. The transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ to world 4-coordinates defines the embedding $z^\mu(\tau, \vec{\sigma})$ of the Riemannian instantaneous 3-spaces Σ_τ into the space-time. By choosing world 4-coordinates centered on the time-like observer, whose world-line is the time axis, we have $x^\mu(\tau) = (x^o(\tau); 0)$: the condition $x^o(\tau) = const.$ is equivalent to $\tau = const.$ and identifies the instantaneous 3-space Σ_τ. If the time-like observer coincides with an asymptotic inertial observer $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\, \tau$ with $\epsilon^\mu_\tau = (1; 0)$, $\epsilon^\mu_r = (0; \delta^i_r)$, $x^\mu_o = (x^o_o; 0)$, then the natural embedding describing the given 3+1 splitting of space-time is $z^\mu(\tau, \sigma^r) = x^\mu_o + \epsilon^\mu_A\, \sigma^A$ and the world 4-metric is ${}^4g_{\mu\nu} = \epsilon^A_\mu\, \epsilon^B_\nu\, {}^4g_{AB}$ (ϵ^A_μ are flat asymptotic cotetrads, $\epsilon^A_\mu\, \epsilon^\mu_B = \delta^A_B$, $\epsilon^A_\mu\, \epsilon^\nu_A = \delta^\mu_\nu$).

From now on we shall denote the curvilinear 3-coordinates σ^r with the notation $\vec{\sigma}$ for the sake of simplicity. Usually the convention of sum over repeated indices is used, except when there are too many summations.

The 4-metric ${}^4g_{AB}$ has signature $\epsilon\,(+---)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions). Flat indices (α), $\alpha = o, a$, are raised and lowered by the flat Minkowski metric ${}^4\eta_{(\alpha)(\beta)} = \epsilon\,(+---)$. We define ${}^4\eta_{(a)(b)} = -\epsilon\, \delta_{(a)(b)}$ with a positive-definite Euclidean 3-metric. On each instantaneous 3-space Σ_τ we have that the 4-metric has a direction-independent limit to the flat Minkowski 4-metric (the asymptotic background) at spatial infinity ${}^4g_{AB}(\tau, \vec{\sigma}) \rightarrow {}^4\eta_{AB(asym)} = \epsilon\,(+---)$. This asymptotic 4-metric allows

[2] It reduces to the special relativistic Poincare' group of the given matter in non-inertial frames of Minkowski space-time when the Newton constant G is switched off.

to define both a flat d'Alambertian $\Box = \partial_\tau^2 - \triangle$ and a flat Laplacian $\triangle = \sum_r \partial_r^2$ on Σ_τ ($\partial_A = \frac{\partial}{\partial \sigma^A}$). We will also need the flat distribution $c(\vec{\sigma}, \vec{\sigma}') = \frac{1}{\Delta}\, \delta^3(\vec{\sigma}, \vec{\sigma}') = -\frac{1}{4\pi}\, \frac{1}{|\vec{\sigma} - \vec{\sigma}'|}$ with $|\vec{\sigma} - \vec{\sigma}'| = \sqrt{\sum_u (\sigma^u - \sigma'^{\,u})^2}$, where $\delta^3(\vec{\sigma}, \vec{\sigma}')$ is the Dirac delta function on the 3-manifold Σ_τ.

In this paper we study the linearization of the Hamilton equations of ADM tetrad gravity in the York canonical basis with the weak field approximation in the family of non-harmonic 3-orthogonal gauges in which the gauge variable (a relativistic inertial effect) York time ${}^3K(\tau, \vec{\sigma})$ is an arbitrary numerical function and the 3-coordinates in Σ_τ are chosen so that the 3-metric is everywhere diagonal. This will be a starting point before facing higher orders in the HPM expansion defined at the end of Section III (at higher orders regularization problems may arise).

Since the gravitational gauge freedom is fixed, we get well defined Hamilton equations for the matter. In particular this *avoids* the introduction of "ad hoc" Lagrangians for the motion of test particles in the resulting gravitational field as it is usually done.

We will look for solutions of the Hamilton equations *near* Minkowski space-time, so that the matter content of the non-flat space-time is assumed to be restricted by a *ultra-violet (UV) cutoff* M: this avoids the appearance of strong gravitational fields. This UV cutoff will allow us to avoid the *slow motion approximation and the Post-Newtonian (PN) expansions* for the N point particles. Naturally our linearization will not be reliable if we look at distances d_i from the $i-th$ particle of the order (or less) of the Schwarzschild gravitational radius $R_M = \frac{2GM}{c^2}$ of the particle, i.e. we must have $d_i > R_M$. Since we have a unique treatment for the near and far zone (we have matter everywhere) we do not need radiative multipoles for the gravitational field but only matter multipoles.We will do a multipolar expansion of the energy-momentum tensor in terms of relativistic Dixon multipoles on the 3-space Σ_τ and we will recover the standard quadrupole emission formula.

An important aspect of the York canonical basis is the separation of the 12 ADM equations for the gravitational field in three sets:

a) the four contracted Bianchi identities for the time derivatives of the unknowns in the super-Hamiltonian and super-momentum constraints (they imply that, if the constraints are solved on the Cauchy surface Σ_{τ_o}, then the solution is respected on the subsequent 3-spaces $\Sigma_{\tau > \tau_o}$);

b) the four equations for the time derivatives of the primary gauge variables (the 3-coordinates and the York time): since we have fixed these gauge variables these equations becomes equations of elliptic type for the lapse and shift functions (the secondary gauge variables $1 + n(\tau, \vec{\sigma})$ and $\bar{n}_{(a)}(\tau, \vec{\sigma})$) of our family of gauges (instead in the harmonic gauges the lapse and shift functions obey wave equations);

c) the Hamilton equations for the tidal variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, $\bar{a} = 1, 2$, which become hyperbolic equations ($\partial_\tau^2\, R_{\bar{a}}(\tau, \vec{\sigma}) = ...$) for $R_{\bar{a}}$ after the elimination of the momenta $\Pi_{\bar{a}}$ by inverting the first half of the Hamilton equations (those for $\partial_\tau\, R_{\bar{a}}$).

To these equations we must add the super-Hamiltonian and super-momentum constraints, which are equations of elliptic type for their unknowns on the 3-space Σ_τ, namely for the conformal factor $\tilde{\phi}(\tau, \vec{\sigma})$ of the 3-metric (namely the 3-volume element) and the off-diagonal terms $\sigma_{(a)(b)}|_{a \not= b}(\tau, \vec{\sigma})$ of the shear of the congruence of Eulerian observers associated with the 3+1 splitting.

All the previous equations and also the matter Hamilton equations are considered on the instantaneous 3-spaces of the whole space-time without making a separation between the *near* and *far or radiation* zones with respect to the particles for the study of gravitational radiation.

As a consequence, only the equations for $\partial_\tau^2 R_{\bar a}(\tau, \vec\sigma)$ will lead to linearized equations involving the flat d'Alambertian and implying matter-dependent *retarded* solutions $R^{(ret)}_{\bar a}(\tau, \vec\sigma)$ (with the homogeneous solutions eliminated by a no-incoming radiation condition with respect to the asymptotic Minkowski 4-metric). We will see that the linearized wave equation implies that only the TT (traceless-transverse) part of the matter stress tensor is relevant. Therefore we recover TT gravitational waves also in this family of non-harmonic gauges. Moreover, by means of a transformation of the 3-coordinates σ^r on the 3-space Σ_τ, we can find a generalized TT gauge with the relativistic inertial effects connected with the York time ${}^3K$ explicitly shown.

Moreover we find the correct energy balance (the back-reaction problem) for the emission of GW's by using the conserved ADM energy and avoiding singular quantities like the gravitational self-forces on the particles [16, 17] due to the Grassmann regularization and objects like the Landau-Lifschitz energy-momentum pseudo-tensor of the gravitational field.

The solutions of the linearized super-Hamiltonian and super-momentum constraints involve only instantaneous quantities on the 3-space Σ_τ and the same happens for the linearized equations for the lapse and shift functions. These are the *instantaneous action-at-a-distance* effects. Moreover the linearized Bianchi identities turn out to be automatically satisfied. Therefore, the linearized solutions for $\tilde\phi$, $\sigma_{(a)(b)}|_{a\neq b}$, $1+n$, $\bar n_{(a)}$, depend upon matter-dependent instantaneous quantities and upon the instantaneous tidal variables $\Gamma^{(1)}_r = \sum_{\bar a} \gamma_{\bar a r}\, R_{\bar a}$. When we replace $\Gamma^{(1)}_r$ with the retarded solution $\Gamma^{(1)(ret)}_r = \sum_{\bar a} \gamma_{\bar a r}\, R^{(ret)}_{\bar a}$, the previous variables become functions of suitable combinations of *instantaneous* and *retarded* terms all depending only on the matter. At the lowest order in $1/c^2$ the results of the harmonic gauge used in the IAU conventions for the solar system are recovered (if ${}^3K$ is negligible inside the solar system).

At this stage the extrinsic curvature tensor ${}^3K_{rs}(\tau, \vec\sigma)$ is completely determined except for the numerical function describing its trace ${}^3K(\tau, \vec\sigma)$. Therefore, the final determination of the dynamical instantaneous 3-spaces Σ_τ (whose inner 3-curvature is determined by the gravitational waves and matter) associated to the linearized solution requires a choice of the York time: this has to be done by relying on the *observational conventions*, which hide a notion of clock synchronization. Instead in the harmonic gauges one uses the Euclidean instantaneous 3-spaces of the inertial observers of Minkowski space-time, avoiding to look at the extrinsic curvature associated to the harmonic solutions!

Both in the previous solutions and in the resulting Hamilton equations for matter the influence of the relativistic inertial effects connected with the York time ${}^3K$ are explicitly shown. It turns out that at the HPM level all the equations depend on the following spatially non-local function of the York time: ${}^3\mathcal{K} \stackrel{def}{=} \frac{1}{\triangle}\, {}^3K$.

In a third paper [18] we will give the PN expansion at all orders of the matter equations of motion (in absence of the electro-magnetic field) in the slow motion limit resulting from the HPM linearization. There we will evaluate the dependence on the inertial gauge variable York time ${}^3K_{(1)} = \triangle\, {}^3\mathcal{K}_{(1)}$ of geometrical and physical quantities. The relevance for astrophysics and cosmology of these results will be discussed in the third paper, after an analysis

of the relation of the gauge problem in general relativity with the conventions used for the description of matter (extended bodies) in the geocentric (GCRS), barycentric (BCRS(and celestial (ICRS) reference frames. There we will discuss the possibility that dark matter can be simulated with these relativistic inertial effects in a Post-Minkowskian (PM) extension of ICRS.

In Section II we review the Hamilton equations of the gravitational field and of matter in our family of 3-orthogonal gauges, which were given in Appendix C of paper I in the 3-orthogonal gauges defined in that paper.

In Section III we define our linearization scheme and we define the HPM expansion. Also the coordinate transformation connecting the 3-orthogonal gauges to the harmonic ones at the lowest HPM order is defined.

In Section IV we solve the linearized constraints and the equations for the lapse and shift functions in our family of 3-orthogonal gauges. Also the ADM Poincare' generators are given till the second order.

In Section V we give the linearization of the equations of motion for the particles and the electro-magnetic field.

In Section VI we give the linearized second order equations of motion for the tidal variables $R_{\bar{a}}$ and we show that they imply the wave equation (with respect to the asymptotic Minkowski 4-metric) for the TT part ${}^4h^{TT}_{(1)rs}$ of the diagonal 3-metric ${}^4g_{(1)rs}$ on the 3-space Σ_τ. Also a generalized (non 3-orthogonal) TT gauge is identified

In Section VII we study the retarded solution for HPM GW with asymptotic background and we show that the dominant term is the emission quadrupole formula by using a multipolar expansion of the energy-momentum tensor in terms of Dixon multipoles. Also the far field behavior is considered. Then we study the energy balance associated to the emission of GW's. Finally we look at the problem of detection of GW's.

In the Conclusions, after a review of the results and of the problems which will appear at the second HPM order, we discuss the *gauge problem in general relativity*, the dependence upon the observational conventions for the 4-coordinates of the description of matter and the relevance of the York time ${}^3K_{(1)}$ for explaining at least part of dark matter as relativistic inertial effects (to be discussed in the third paper [18]).

For a comparison with our formulation, in Appendix A there is a review of the standard approach to GW's by using Einstein's equations in harmonic gauges.

In Appendix B there is a discussion of Dixon multipoles and of the multipolar expansion of the energy-momentum tensor.

In Appendix C there is the study of the balance for momentum and angular momentum in GW's emission.

II. THE HAMILTON EQUATIONS IN THE 3-ORTHOGONAL SCHWINGER TIME GAUGES

As shown in paper I, the 3-orthogonal gauges of ADM tetrad gravity are the family of Schwinger time gauges where we have

$$\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0, \qquad \varphi_{(a)}(\tau,\vec{\sigma}) \approx 0,$$

$$\theta^i(\tau,\vec{\sigma}) \approx 0, \qquad \pi_{\tilde{\phi}}(\tau,\vec{\sigma}) = \frac{c^3}{12\pi\, G}\, {}^3K(\tau,\vec{\sigma}) \approx \frac{c^3}{12\pi\, G}\, F(\tau,\vec{\sigma}), \tag{2.1}$$

in the York canonical basis ($a, r = 1,2,3$, $\bar{a} = 1,2$)

$\varphi_{(a)}$	$\alpha_{(a)}$	n	$\bar{n}_{(a)}$	θ^r	$\tilde{\phi}$	$R_{\bar{a}}$
$\pi_{\varphi_{(a)}} \approx 0$	$\pi^{(\alpha)}_{(a)} \approx 0$	$\pi_n \approx 0$	$\pi_{\bar{n}_{(a)}} \approx 0$	$\pi^{(\theta)}_r$	$\pi_{\tilde{\phi}}$	$\Pi_{\bar{a}}$

(2.2)

The tidal variables of the gravitational field are $R_{\bar{a}}$, $\Pi_{\bar{a}}$. The primary gauge variables (the inertial effects connected with the choice of 3-coordinates on Σ_τ and with clock synchronization) are θ^r and $\pi_{\tilde{\phi}}$: their conjugate variables $\pi^{(\theta)}_r$ and $\tilde{\phi}$ are the unknowns in the super-momentum and super-Hamiltonian constraints, respectively. While $\tilde{\phi}(\tau,\vec{\sigma})$ is the conformal factor of the 3-metric (namely the 3-volume element), $\pi^{(\theta)}_r$ may be replaced with the off-diagonal terms $\sigma_{(a)(b)}|_{a\neq b}(\tau,\vec{\sigma})$ of the shear of the congruence of Eulerian observers associated with the 3+1 splitting (the diagonal elements are connected to the tidal momenta $\Pi_{\bar{a}}$). The secondary gauge (inertial) variables (determined after a gauge fixation of the primary ones) are the shift ($n^r = {}^3\bar{e}^r_{(a)}\, \bar{n}_{(a)}$) and lapse ($1+n$) functions.

The canonical variables for the particles are $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$, $i = 1,..,N$ (we use the notation κ_{ir} instead of the one $\check{\kappa}_{ir}$ of paper I). For the electromagnetic field in the radiation gauge the canonical variables are $A_{\perp r}(\tau,\vec{\sigma})$, $\pi^r_\perp(\tau,\vec{\sigma})$ and from paper I we have the following expression for the electro-magnetic fields $F_{rs} = \partial_r\, A_{\perp s} - \partial_s\, A_{\perp r}$, $B_r = \epsilon_{ruv}\, \partial_u\, A_{\perp v}$, $E_r = -F_{\tau r} = -\partial_\tau\, A_{\perp r} + \partial_r\, A_\tau$ with A_τ given in Eq.(3.32) of I.

The members of this family of gauges differ for the value of the York time ${}^3K(\tau,\vec{\sigma}) \approx F(\tau,\vec{\sigma})$, where $F(\tau,\vec{\sigma})$ is a numerical function (for $F(\tau,\vec{\sigma}) = const.$ we get the constant mean curvature (CMC) gauges of ADM theory). This gauge variable, describing a relativistic inertial effect (which does not exist in Newton theory in Galileo space-time), is the remnant of the special relativistic gauge freedom in choosing the convention for clock synchronization, i.e. for the identification of the instantaneous 3-spaces Σ_τ. While in special relativity the whole extrinsic curvature tensor ${}^3K_{rs}(\tau,\vec{\sigma})$ of Σ_τ is pure gauge, in general relativity it is determined by the dynamics with the exception of its trace [3].

In this Section we will give the restriction to this family of gauges of the Hamilton equations and of the constraints given in paper I by using the results of its Appendix C.

[3] Let us remark that positive (negative) extrinsic curvature implies that the instantaneous 3-space, as an embedded 3-manifold, is a concave (convex) 3-surface of the space-time

We will use the following notational conventions (most of them are defined in paper I):

a) We write the conformal factor of the 3-metric in the form $\phi(\tau, \vec{\sigma}) = \tilde{\phi}^{1/6}(\tau, \vec{\sigma}) = e^{q(\tau, \vec{\sigma})}$. Then we have $\phi^{-1}\, \partial_r\, \phi = \frac{1}{6}\, \tilde{\phi}^{-1}\, \partial_r\, \tilde{\phi} = \partial_r\, q$, $\phi^{-1}\, \partial_r^2\, \phi = \partial_r^2\, q + (\partial_r\, q)^2$ (this notation was not used in paper I).

b) We use V_{ra} for $V_{ra}(\theta^n)$ to simplify the notation. We use the notation $V_{ra}(0) = \delta_{ra}$, $V_{(i)ra} = \frac{\partial\, V_{ra}(\theta^n)}{\partial\, \theta^i}|_{\theta^i=0}$, $B_{(i)jw} = \frac{\partial\, B_{jw}(\theta^n)}{\partial\, \theta^i}|_{\theta^i=0}$. As said in Subsection IIC of paper I, we use angles θ^i corresponding to canonical coordinates of first kind on the group manifold of SO(3), because this implies $V_{(i)rs} = 2\, B_{(i)rs} = \epsilon_{irs}$.

c) The set of numerical parameters $\gamma_{\bar{a}a}$ satisfies [6, 7] $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u}\, \gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}$, $\sum_{\bar{a}} \gamma_{\bar{a}u}\, \gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. A different York canonical basis is associated to each solution of these equations.

As shown in paper I, in the York canonical basis we have the following building blocks for the Einstein-Maxwell- Particle system in the radiation gauge of the electro-magnetic field [4] (${}^3\bar{e}_{(a)r}$ and ${}^3\bar{e}^r_{(a)}$ are cotriads and triads on Σ_τ respectively)

$$
\begin{aligned}
{}^4g_{\tau\tau} &= \epsilon\, \Big[(1+n)^2 - \sum_a \bar{n}^2_{(a)}\Big],\\
{}^4g_{\tau r} &= -\epsilon\, \sum_a \bar{n}_{(a)}\, {}^3\bar{e}_{(a)r} = -\epsilon\, \tilde{\phi}^{1/3}\, Q_r\, \bar{n}_{(r)},\\
{}^4g_{rs} &= -\epsilon\, {}^3g_{rs} = -\epsilon\, \sum_a {}^3\bar{e}_{(a)r}\, {}^3\bar{e}_{(a)s} = -\epsilon\, \phi^4\, {}^3\hat{g}_{rs} \;=\; -\epsilon\, \tilde{\phi}^{2/3}\, Q_r^2\, \delta_{rs},
\end{aligned}
$$

$$
Q_a \;=\; e^{\Gamma_q^{(1)}}, \quad \Gamma_a^{(1)} = \sum_{\bar{a}}^{1,2} \gamma_{\bar{a}a}\, R_{\bar{a}}, \qquad \tilde{\phi} = \phi^6 = e^{6\, q} = \sqrt{\gamma} = \sqrt{det\, {}^3g} = {}^3\bar{e},
$$

$$
{}^3e_{(a)r} \;=\; {}^3\bar{e}_{(a)r} = \tilde{\phi}^{1/3}\, Q_a\, \delta_{ra}, \qquad {}^3e^r_{(a)} = {}^3\bar{e}^r_{(a)} = \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \delta_{ra},
$$

$$
\begin{aligned}
\pi_i^{(\theta)} &= \frac{c^3}{8\pi\, G}\, \tilde{\phi}\, \sum_{ab} Q_a\, Q_b^{-1}\, \epsilon_{iab}\, \sigma_{(a)(b)}|_{a\neq b},\\
&\qquad \tilde{\phi}\, \sigma_{(a)(b)}|_{a\neq b} = -\frac{8\pi\, G}{c^3}\, \frac{\epsilon_{abi}}{Q_b\, Q_a^{-1} - Q_a\, Q_b^{-1}}\, \pi_i^{(\theta)},\\
\Pi_{\bar{a}} &= -\frac{c^3}{8\pi\, G}\, \tilde{\phi}\, \sum_a \gamma_{\bar{a}a}\, \sigma_{(a)(a)}, \qquad \tilde{\phi}\, \sigma_{(a)(a)} = -\frac{8\pi\, G}{c^3}\, \sum_{\bar{a}} \gamma_{\bar{a}a}\, \Pi_{\bar{a}},
\end{aligned}
$$

[4] The so-called gothic inverse 4-metric $\mathbf{h}^{AB} = \sqrt{-{}^4g}\, {}^4g^{AB} - {}^4\eta^{AB}$ takes the form: $\mathbf{h}^{\tau\tau} = \epsilon\, (\frac{\tilde{\phi}}{1+n} - 1)$, $\mathbf{h}^{\tau r} = -\epsilon\, \frac{\tilde{\phi}^{2/3}}{1+n}\, Q_r^{-1}\, \bar{n}_{(r)}$, $\mathbf{h}^{rs} = -\epsilon\, [\frac{\tilde{\phi}^{1/3}}{1+n}\, Q_r^{-1}\, Q_s^{-1}\, ((1+n)^2\, \delta_{rs} - \bar{n}_{(r)}\, \bar{n}_{(s)}) - \delta_{rs}]$. At spatial infinity we get $\mathbf{h}^{AB} \to 0$.

$$
\begin{aligned}
{}^3K_{rs} &\approx -\frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1/3}\, \Big(Q_r^2\, \delta_{rs}\, [2 \sum_{\bar b} \gamma_{\bar b r}\, \Pi_{\bar b} - \tilde{\phi}\, \pi_{\tilde{\phi}}] - 2\, Q_r\, Q_s\, \frac{\epsilon_{rsi}\, \pi_i^{(\theta)}}{Q_r\, Q_s^{-1} - Q_s\, Q_r^{-1}} \Big) = \\
&= \tilde{\phi}^{2/3}\, \Big[\frac{4\pi\, G}{c^3}\, \pi_{\tilde{\phi}}\, Q_r^2\, \delta_{rs} + (1 - \delta_{rs})\, \sigma_{(r)(s)}\, Q_r\, Q_s - \frac{8\pi\, G}{c^3}\, \tilde{\phi}^{-1} \sum_{\bar a} \gamma_{\bar a r}\, \Pi_{\bar a}\, Q_r^2\, \delta_{rs} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &\stackrel{def}{=} \frac{1}{2} \sum_{ab} \sigma_{(a)(b)}\, \sigma_{(a)(b)} = \frac{1}{2}\, \Big(\frac{8\pi\, G}{c^3} \Big)^2\, \tilde{\phi}^{-2}\, \Big[\sum_{\bar a} \Pi_{\bar a}^2 + \\
&+ 2\, \Big(\frac{(\pi_1^{(\theta)})^2}{(Q_2\, Q_3^{-1} - Q_3\, Q_2^{-1})^2} + \frac{(\pi_2^{(\theta)})^2}{(Q_3\, Q_1^{-1} - Q_1\, Q_3^{-1})^2} + \frac{(\pi_3^{(\theta)})^2}{(Q_1\, Q_2^{-1} - Q_2\, Q_1^{-1})^2} \Big) \Big]. \quad (2.3)
\end{aligned}
$$

Eqs.(2.18) and (6.2) of I have been used. $\theta = -\epsilon\, {}^3K$ and $\sigma_{(a)(b)}$ are the expansion and the shear of the congruence of Eulerian observers of Σ_τ as shown in paper I. In paper I it is also shown that the original momenta conjugate to the cotriads ${}^3e_{(a)r}$ before going to the York canonical basis are ${}^3\pi^r_{(a)} \approx \tilde{\phi}^{-1/3}\, [\delta_{ra}\, Q_a^{-1}\, (\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar a} \gamma_{\bar a a}\, \Pi_{\bar a}) + \sum_i Q_r^{-1}\, \frac{\epsilon_{ari}\, \pi_i^{(\theta)}}{Q_r\, Q_a^{-1} - Q_a\, Q_r^{-1}}]$.

From Eqs.(6.4) of I we have the following expressions of the mass and momentum densities (we use the notation $\mathcal{M}$ and κ_{ir} instead of $\check{\mathcal{M}}$ and $\check{\kappa}_{ir}$ of I for the quantities in the electromagnetic radiation gauge; see the Introduction for $c(\vec{\sigma}, \vec{\sigma}^{'})$)

$$
\begin{aligned}
\mathcal{M}(\tau, \vec{\sigma})|_{\theta^i=0} &= \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau))\, \eta_i\, \sqrt{m_i^2\, c^2 + \tilde{\phi}^{-2/3} \sum_a Q_a^{-2}\, \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a} \Big)^2}\, (\tau, \vec{\sigma}) + \\
&+ \frac{1}{2c}\, \Big[\tilde{\phi}^{-1/3}\, \Big(\sum_{rsa} Q_a^2\, \delta_{ra}\, \delta_{sa}\, \pi^r_\perp\, \pi^s_\perp + \frac{1}{2} \sum_{ab} Q_a^{-2}\, Q_b^{-2}\, F_{ab}\, F_{ab} \Big) \Big] (\tau, \vec{\sigma}) - \\
&- \frac{1}{2c}\, \Big[\tilde{\phi}^{-1/3} \sum_{rsan} Q_a^2\, \delta_{ra}\, \delta_{sa}\, \Big(2\, \pi^r_\perp - \sum_m \delta^{rm} \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^m} \Big) \\
&\quad \delta^{sn} \sum_j Q_j\, \eta_j\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial\, \sigma^n} \Big] (\tau, \vec{\sigma}), \\
\mathcal{M}_r(\tau, \vec{\sigma}) &= \sum_{i=1}^N \eta_i\, \Big(\kappa_{ir}(\tau) - \frac{Q_i}{c}\, A_{\perp r}(\tau, \vec{\sigma}) \Big)\, \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) - \\
&- \frac{1}{c} \sum_s F_{rs}(\tau, \vec{\sigma})\, \Big(\pi^s_\perp(\tau, \vec{\sigma}) - \sum_n \delta^{sn} \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^n} \Big),
\end{aligned}
\quad (2.4)
$$

From Eqs. (C8), (C12) and (C20) of I we get the following expressions for the Laplace-Beltrami operator and for the intrinsic 3-curvature ${}^3R[\theta^n, \phi, R_{\bar a}] = \phi^{-5}\, \Big(- 8\, \hat{\triangle}\, \phi + {}^3\hat{R}\, \phi \Big) = \phi^{-6}\, \Big(\mathcal{S} + \mathcal{T} \Big)$ with ${}^3\hat{R}[\theta^n, R_{\bar a}] = \phi^{-2}\, \Big(\mathcal{S} + \mathcal{T}_1 \Big)$

$$
\begin{aligned}
\hat{\triangle}|_{\theta^i=0} &= \sum_a Q_a^{-2}(\tau,\vec{\sigma})\,\Big[\partial_a^2 - 2\sum_{\bar{b}} \gamma_{\bar{b}a}\,\partial_a\,R_{\bar{b}}(\tau,\vec{\sigma})\,\partial_a\Big] = \\
&= \sum_a Q_a^{-2}(\tau,\vec{\sigma})\,\Big[\partial_a^2 - 2\,\partial_a\,\Gamma_a^{(1)}(\tau,\vec{\sigma})\,\partial_a\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}|_{\theta^i=0} &= \tilde{\phi}^{1/3}\sum_a Q_a^{-2}\Big(\sum_{\bar{b}}\Big[\sum_{\bar{c}}(2\,\gamma_{\bar{b}a}\,\gamma_{\bar{c}a} - \delta_{\bar{b}\bar{c}})\,\partial_a\,R_{\bar{b}}\,\partial_a\,R_{\bar{c}} - \\
&- 4\,\gamma_{\bar{b}a}\,\partial_a\,q\,\partial_a\,R_{\bar{b}}\Big] + 8\,(\partial_a\,q)^2\Big),\\
\mathcal{T}(\tau,\vec{\sigma})|_{\theta^i=0} &= -2\Big[\tilde{\phi}^{1/3}\sum_a Q_a^{-2}\Big(4\Big[\partial_a^2\,q + 2\,(\partial_a\,q)^2\Big] - \\
&- \partial_a^2\,\Gamma_a^{(1)} - 2\,(5\,\partial_a\,q - \partial_a\,\Gamma_a^{(1)})\,\partial_a\,\Gamma_a^{(1)}\Big)\Big](\tau,\vec{\sigma}),\\
\mathcal{T}_1(\tau,\vec{\sigma})|_{\theta^i=0} &= -2\Big[\tilde{\phi}^{1/3}\sum_a Q_a^{-2}\Big(4\,(\partial_a\,q)^2 - \partial_a^2\,\Gamma_a^{(1)} - 2\,(\partial_a\,q - \partial_a\,\Gamma_a^{(1)})\,\partial_a\,\Gamma_a^{(1)}\Big)\Big](\tau,\vec{\sigma}),\\
{}^3\hat{R}(\tau,\vec{\sigma})|_{\theta^i=0} &= \sum_a\Big(Q_a^{-2}\Big[2\,\partial_a^2\,\Gamma_a^{(1)} - \sum_{\bar{b}}(\partial_a\,R_{\bar{b}})^2 - 2\,\Big(\partial_a\,\Gamma_a^{(1)}\Big)^2\Big]\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{2.5}
$$

In the 3-orthogonal gauges we have the following form of the constraints, of the Bianchi identities and of the equations of motion of gauge and physical variables.

A. The Constraints

The super-Hamiltonian and super-momentum constraints determining $\phi = \tilde{\phi}^{1/6} = e^q$ and $\sigma_{(a)(b)}|_{a\neq b} = -\frac{8\pi\,G}{c^3}\,\tilde{\phi}^{-1}\,\sum_i \frac{\epsilon_{abi}\,\pi_i^{(\theta)}}{Q_b\,Q_a^{-1} - Q_a\,Q_b^{-1}}$, given in Eqs. (6.5), (6.6), (C12), (C20) of I, are [Eqs.(2.4) and (2.5) are also needed; we use the notation $\mathcal{H}_{(a)}$ instead of $\bar{\tilde{\mathcal{H}}}_{(a)}$ of I]

$$
\begin{aligned}
\mathcal{H}(\tau,\vec{\sigma})|_{\theta^i=0} &= \frac{c^3}{2\pi\,G}\,\phi(\tau,\vec{\sigma})\,\Big((\hat{\triangle}|_{\theta^i=0} - \frac{1}{8}\,{}^3\hat{R}|_{\theta^i=0})\,\phi + \frac{2\pi\,G}{c^3}\,\phi^{-1}\,\mathcal{M}|_{\theta^i=0} + \\
&+ \frac{8\,\pi^2\,G^2}{c^6}\,\phi^{-7}\sum_{\bar{a}}\Pi_{\bar{a}}^2 + \frac{1}{8}\,\phi^5\sum_{ab,a\neq b}\sigma_{(a)(b)}\,\sigma_{(a)(b)} - \frac{1}{12}\,\phi^5\,({}^3K)^2\Big)(\tau,\vec{\sigma}) \approx 0,
\end{aligned}
\tag{2.6}
$$

$$\mathcal{H}_{(a)}|_{\theta^i=0}(\tau,\vec{\sigma}) \;=\; -\frac{c^3}{8\pi\,G}\,\tilde{\phi}^{2/3}(\tau,\vec{\sigma})\Big(\sum_{b\neq a} Q_b^{-1}\Big[\partial_b\,\sigma_{(a)(b)} + \Big(6\,\partial_b\,q + \partial_b\,(\Gamma_a^{(1)} - \Gamma_b^{(1)})\Big)\,\sigma_{(a)(b)}\Big] -$$
$$-\; Q_a^{-1}\Big[\frac{2}{3}\,\partial_a\,{}^3K + \frac{8\pi\,G}{c^3}\,\tilde{\phi}^{-1}\Big(\sum_{\bar{b}}(\gamma_{\bar{b}a}\,\partial_a\,\Pi_{\bar{b}} - \partial_a\,R_{\bar{b}}\,\Pi_{\bar{b}}) + \mathcal{M}_{(a)}\Big)\Big]\Big)(\tau,\vec{\sigma}) \quad\approx 0. \tag{2.7}$$

All the constraints depend on the York time ${}^3K$.

B. The Contracted Bianchi Identities

The Hamilton equations for the unknowns in the constraints, given in Eqs.(6.7), (6.8), (C21), (C11), (C19) and (C5) of I, are ($\circeq$ means evaluated by means of the equations of motion)

$$\partial_\tau\,\tilde{\phi}(\tau,\vec{\sigma})|_{\theta^i=0} = 6\,(\tilde{\phi}\,\partial_\tau\,q)(\tau,\vec{\sigma})|_{\theta^i=0} \;\circeq\; \Big[-(1+n)\,\tilde{\phi}\,{}^3K + \tilde{\phi}^{2/3}\sum_a Q_a^{-1}\Big(\partial_a\,\bar{n}_{(a)} +$$
$$+\;\bar{n}_{(a)}\Big(4\,\partial_a\,q - \partial_a\,\Gamma_a^{(1)}\Big)\Big)\Big](\tau,\vec{\sigma}), \tag{2.8}$$

$$\partial_\tau\,\pi_i^{(\theta)}(\tau,\vec{\sigma})|_{\theta^i=0} \;=\; \frac{c^3}{8\pi\,G}\,\tilde{\phi}\sum_{ab}\epsilon_{iab}\,Q_a\,Q_b^{-1}\Big[\partial_\tau\,\sigma_{(a)(b)} + \Big(6\,\partial_\tau\,q + \partial_\tau\,(\Gamma_a^{(1)} - \Gamma_b^{(1)})\Big)\,\sigma_{(a)(b)}\Big]. \tag{2.9}$$

We can extract $\partial_\tau\,\sigma_{(a)(b)}$ from Eqs.(2.9) due to Eqs.(2.3), i.e. $\frac{8\pi\,G}{c^3}\,\pi_i^{(\theta)} = \tilde{\phi}\sum_{ab}\epsilon_{iab}\,Q_a\,Q_b^{-1}\,\sigma_{(a)(b)}$. By using $V_{(i)rs} = 2\,B_{(i)rs} = \epsilon_{irs}$ and Eqs. (6.8) and (C5), (C11), (C19), (C21) of I, we get

$$\partial_\tau\,\sigma_{(a)(b)}|_{a\neq b,\theta^i=0} \;\circeq\; -\Big[\Big(6\,\partial_\tau\,q + \frac{Q_b\,Q_a^{-1} + Q_a\,Q_b^{-1}}{Q_b\,Q_a^{-1} - Q_a\,Q_b^{-1}}\,\partial_\tau\,(\Gamma_a^{(1)} - \Gamma_b^{(1)})\Big)\,\sigma_{(a)(b)}|_{a\neq b}\Big](\tau,\vec{\sigma}) -$$
$$-\;\Big(\tilde{\phi}^{-1}\sum_i\frac{\epsilon_{abi}}{Q_b\,Q_a^{-1} - Q_a\,Q_b^{-1}}\Big)(\tau,\vec{\sigma})\,\Big[F_i(\tau,\vec{\sigma}) -$$
$$-\;\int d^3\sigma_1\,\Big((1+n(\tau,\vec{\sigma}_1))\,\Big[\frac{8\pi\,G}{c^3}\,\frac{\delta\,\mathcal{M}(\tau,\vec{\sigma}_1)}{\delta\,\theta^i(\tau,\vec{\sigma})}|_{\theta^i=0} - \frac{1}{2}\,\frac{\delta\,\mathcal{S}(\tau,\vec{\sigma}_1)}{\delta\,\theta^i(\tau,\vec{\sigma})}|_{\theta^i=0}\Big] -$$
$$-\;\frac{1}{2}\,n(\tau,\vec{\sigma}_1)\,\frac{\delta\,\mathcal{T}(\tau,\vec{\sigma}_1)}{\delta\,\theta^i(\tau,\vec{\sigma})}|_{\theta^i=0}\Big)\Big],$$

$$
\begin{aligned}
&\int d^3\sigma_1\, \Big(1 + n(\tau, \vec{\sigma}_1)\Big)\, \frac{\delta\, \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta\, \theta^i(\tau, \vec{\sigma})}|_{\theta^i=0} = \\
&= \frac{1}{2} \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau))\, \eta_i \Bigg((1+n) \\
&\frac{\tilde{\phi}^{-2/3} \sum_{rsa} Q_a^{-2}\, (V_{(i)ra}\, \delta_{sa} + \delta_{ra}\, V_{(i)sa}) \Big(\kappa_{ir}(\tau) - \frac{Q_i}{c}\, A_{\perp\, r}\Big) \Big(\kappa_{is}(\tau) - \frac{Q_i}{c}\, A_{\perp\, s}\Big)}{\sqrt{m_i^2\, c^2 + \tilde{\phi}^{-2/3} \sum_a Q_a^{-2} \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}}\Bigg)(\tau, \vec{\sigma}) + \\
&+ \Big(1 + n(\tau, \vec{\sigma})\Big) \Big(\tilde{\phi}^{-1/3}\, \frac{1}{c} \Big[\sum_{ars} Q_a^2\, V_{(i)ra}\, \delta_{sa} \pi_\perp^r\, \pi_\perp^s + \sum_{abr} Q_a^{-2}\, Q_b^{-2}\, V_{(i)ra}\, F_{rb}\, F_{ab} - \\
&- \frac{1}{2} \sum_{arsn} Q_a^2 \Big(V_{(i)ra}\, \delta_{sa} + V_{(i)sa}\, \delta_{ra}\Big) \Big(2\, \pi_\perp^r - \sum_m \delta^{rm} \sum_i Q_i\, \eta_i\, \partial_m\, c(\vec{\sigma}, \vec{\eta}_i(\tau))\Big) \\
&\delta^{sn} \sum_j Q_j\, \eta_j\, \partial_n\, c(\vec{\sigma}, \vec{\eta}_j(\tau))\Big]\Big)(\tau, \vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&\int d^3\sigma_1\, [1 + n(\tau, \vec{\sigma}_1)]\, \frac{\delta\, \mathcal{S}(\tau, \vec{\sigma}_1)}{\delta\, \theta^i(\tau, \vec{\sigma})}\, |_{\theta^i=0} = \\
&= -2 \Big[\tilde{\phi}^{1/3} \sum_{ra} Q_a^{-2}\, V_{(i)ra} \Big(\partial_a\, n\, \partial_r\, q - \sum_{\bar{b}} \gamma_{\bar{b}a}\, \partial_r\, R_{\bar{b}}) + \partial_r\, n\, \partial_a\, q - \sum_{\bar{b}} \gamma_{\bar{b}r}\, \partial_a\, R_{\bar{b}}) - \\
&- (1+n) \Big[2\, (2\, \partial_a\, q\, \partial_r\, q - \partial_a\, \partial_r\, q) + \sum_{\bar{b}} (\gamma_{\bar{b}a} + \gamma_{\bar{b}r})\, \partial_a\, \partial_r\, R_{\bar{b}} + \\
&+ 2 \sum_{\bar{b}} (\gamma_{\bar{b}a}\, \partial_a\, q\, \partial_r\, R_{\bar{b}} + \gamma_{\bar{b}r}\, \partial_r\, q\, \partial_a\, R_{\bar{b}}) - \\
&- \sum_{\bar{b}\bar{c}} (2\, \gamma_{\bar{b}r}\, \gamma_{\bar{c}a} + \delta_{\bar{b}\bar{c}})\, \partial_a\, R_{\bar{b}}\, \partial_r\, R_{\bar{c}}\Big]\Big)\Big](\tau, \vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&\int d^3\sigma_1\, n(\tau, \vec{\sigma}_1)\, \frac{\delta\, \mathcal{T}(\tau, \vec{\sigma}_1)}{\delta\, \theta^i(\tau, \vec{\sigma})}|_{\theta^i=0} = \\
&= -2 \Big[\tilde{\phi}^{1/3} \sum_{ra} Q_a^{-2}\, V_{(i)ra} \Big(\partial_r\, \partial_a\, n - 3\, (\partial_r\, n\, \partial_a\, q + \partial_a\, n\, \partial_r\, q)\Big)\Big](\tau, \vec{\sigma}),
\end{aligned}
$$

$$\begin{aligned}
F_i(\tau,\vec{\sigma}) &= \frac{8\pi\,G}{c^3}\int d^3\sigma_1\,\sum_a \bar{n}_{(a)}(\tau,\vec{\sigma}_1)\,\frac{\delta\,\mathcal{H}_{(a)}(\tau,\vec{\sigma}_1)}{\delta\,\theta^i(\tau,\vec{\sigma})}\,|_{\theta^i=0} -\\
&- \frac{3}{2}\Big((1+n)\,\tilde{\phi}\,\sum_{m\neq n}\frac{\sigma_{(m)(n)}}{Q_m\,Q_n^{-1}-Q_n\,Q_m^{-1}}\sum_{c\neq d}\sum_{tk}\epsilon_{mnt}\,\epsilon_{ikt}\,\epsilon_{kcd}\,Q_c\,Q_d^{-1}\,\sigma_{(c)(d)}\Big)(\tau,\vec{\sigma}) =\\
&= -\Big[\tilde{\phi}^{-1/3}\sum_a\,\Big(\sum_r\partial_r\,\bar{n}_{(a)}\,\Big[\frac{8\pi\,G}{c^3}\,Q_a^{-1}\,\epsilon_{ira}\,\sum_{\bar{b}}(\gamma_{\bar{b}a}-\gamma_{\bar{b}r})\,\Pi_{\bar{b}} -\\
&- \tilde{\phi}\,\Big(\sum_{b\neq a}Q_b^{-1}\,\epsilon_{irb}\,\sigma_{(a)(b)} + Q_a^{-1}\sum_{b\neq r}Q_b\,Q_r^{-1}\,\epsilon_{iab}\,\sigma_{(r)(b)}\Big)\Big] +\\
&+ \tilde{\phi}\,\partial_a\,\bar{n}_{(a)}\,Q_a^{-1}\sum_{b\neq c}\epsilon_{icb}\,Q_b\,Q_c^{-1}\,\sigma_{(b)(c)}\Big)\Big](\tau,\vec{\sigma}) +\\
&+ \Big[\tilde{\phi}^{-1/3}\sum_a\bar{n}_{(a)}\,\Big(\frac{8\pi\,G}{c^3}\,Q_a^{-1}\sum_b\epsilon_{iba}\,\Big[(\frac{c^3}{12\pi\,G}\,\tilde{\phi}\,\partial_b\,{}^3K - \sum_{\bar{b}}\partial_b\,R_{\bar{b}}\,\Pi_{\bar{b}}) +\\
&+ \sum_{\bar{b}}\gamma_{\bar{b}b}\,\partial_b\,\Pi_{\bar{b}} + \sum_{\bar{b}}(\gamma_{\bar{b}a}-\gamma_{\bar{b}b})\,(2\,\partial_b\,q + \partial_b\,\Gamma_a^{(1)})\,\Pi_{\bar{b}} + \mathcal{M}_b\Big] +\\
&\quad + \tilde{\phi}\,\Big[Q_a^{-1}\sum_{b,c\neq b}Q_b\,Q_c^{-1}\Big(\epsilon_{iab}\,\partial_c\,\sigma_{(b)(c)} - \epsilon_{icb}\,\partial_a\,\sigma_{(b)(c)} + 4\,(\epsilon_{iab}\,\partial_c\,q -\\
&\quad - \epsilon_{icb}\,\partial_a\,q)\,\sigma_{(b)(c)} - (\epsilon_{iab}\,\partial_c - \epsilon_{icb}\,\partial_a)\,(\Gamma_a^{(1)} - \Gamma_b^{(1)} + \Gamma_c^{(1)})\,\sigma_{(b)(c)}\Big) -\\
&\quad - \sum_{r\,b\neq a}Q_b^{-1}\,\sigma_{(a)(b)}\,\epsilon_{irb}\,(2\,\partial_r\,q + \partial_r\,\Gamma_a^{(1)})\Big]\Big)\Big](\tau,\vec{\sigma}).
\end{aligned} \tag{2.10}$$

C. The Shift Functions

From the τ-preservation of the gauge fixings $\theta^i(\tau,\vec{\sigma})\approx 0$, see Eqs.(6.10) of I, we get the following equations for the shift functions

$$\begin{aligned}
&\Big(Q_b^{-1}\,\partial_b\,\bar{n}_{(a)} + Q_a^{-1}\,\partial_a\,\bar{n}_{(b)} - \Big[Q_b^{-1}\,\Big(2\,\partial_b\,q + \partial_b\,\Gamma_a^{(1)}\Big)\,\bar{n}_{(a)} +\\
&+ Q_a^{-1}\,\Big(2\,\partial_a\,q + \partial_a\,\Gamma_b^{(1)}\Big)\,\bar{n}_{(b)}\Big]\Big)(\tau,\vec{\sigma}) \approx\\
&\approx 2\,\Big[\tilde{\phi}^{1/3}\,(1+n)\,\sigma_{(a)(b)}|_{a\neq b}\Big](\tau,\vec{\sigma}),\qquad a\neq b.
\end{aligned} \tag{2.11}$$

D. The Instantaneous 3-Space and the Lapse Functions

The preservation in τ of the gauge fixing constraint ${}^3K(\tau,\vec{\sigma})\approx F(\tau,\vec{\sigma})$ given in Eqs. (2.1) gives Eq.(6.12) of paper I. The restriction of this equation to our family of gauges (by using

Eqs. (C22), (C10) and (C17) of I and the super-Hamiltonian constraint) gives the following equation for the determination of the lapse function (it is the Raychaudhuri equation)

$$
\begin{aligned}
&\Big(\sum_a Q_a^{-1} \Big[\partial_a^2 n + \frac{1}{2} \Big(2\, \partial_a q - \frac{7}{2}\, \partial_a \Gamma_a^{(1)} \Big)\, \partial_a n \Big] - \\
&- \frac{4\pi G}{c^3}\, (1+n) \Big[\tilde{\phi}^{-1/3}\, \mathcal{M} - \frac{1}{2}\, \tilde{\phi}^{-1/6} \int d^3\sigma_1 \Big(1 + n(\tau, \vec{\sigma}_1) \Big) \frac{\delta\, \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta\, \phi(\tau, \vec{\sigma})} \Big] - \\
&- (1+n) \Big[\frac{1}{3}\, \tilde{\phi}^{2/3}\, ({}^3K)^2 + (\frac{8\pi G}{c^3})^2\, \tilde{\phi}^{-4/3} \sum_{\bar{a}} \Pi_{\bar{a}}^2 + \tilde{\phi}^{2/3} \sum_{ab, a \neq b} \sigma_{(a)(b)}^2 \Big] - \\
&- \tilde{\phi}^{2/3} \Big[- \partial_\tau\, {}^3K + \tilde{\phi}^{-1/3} \sum_a \bar{n}_{(a)}\, Q_a^{-1}\, \partial_a\, {}^3K \Big] \Big)(\tau, \vec{\sigma}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\int d^3\sigma_1 \Big(1 + n(\tau, \vec{\sigma}_1) \Big) \frac{\delta\, \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta\, \phi(\tau, \vec{\sigma})} = \\
&= -2 \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau))\, \eta_i \Big((1+n)\, \frac{\tilde{\phi}^{-5/6} \sum_a Q_a^{-2} \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a} \Big)^2}{\sqrt{m_i^2\, c^2 + \tilde{\phi}^{-2/3} \sum_a Q_a^{-2} \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a} \Big)^2}} \Big)(\tau, \vec{\sigma}) - \\
&- \Big(1 + n(\tau, \vec{\sigma}) \Big) \Big(\tilde{\phi}^{-1/2} \Big[\frac{1}{c} \sum_{ars} Q_a^2\, \delta_{ra}\, \delta_{sa}\, \pi_\perp^r\, \pi_\perp^s + \frac{1}{2c} \sum_{ab} Q_a^{-2}\, Q_b^{-2}\, F_{ab}\, F_{ab} - \\
&- \frac{1}{c} \sum_{arsn} Q_a^2\, \delta_{ra}\, \delta_{sa} \Big(2\, \pi_\perp^r - \sum_m \delta^{rm} \sum_i Q_i\, \eta_i\, \partial_m\, c(\vec{\sigma}, \vec{\eta}_i(\tau)) \Big) \\
&\quad \delta^{sn} \sum_j Q_j\, \eta_j\, \partial_n\, c(\vec{\sigma}, \vec{\eta}_j(\tau)) \Big] \Big)(\tau, \vec{\sigma}).
\end{aligned}
\tag{2.12}
$$

We can also consider gauges in which in Eq.(2.1) we have ${}^3K(\tau, \vec{\sigma}) \approx F(\tau, \vec{\sigma}, \vec{\sigma} - \vec{\eta}_1(\tau), ..., \vec{\sigma} - \vec{\eta}_N(\tau))$. In this case we have $\partial_\tau\, {}^3K = \partial_\tau\, F|_{\vec{\eta}_i(\tau)} + \sum_i \frac{\partial F}{\partial \vec{\eta}_i}\, \dot{\vec{\eta}}_i(\tau)$ with $\dot{\vec{\eta}}_i(\tau)$ given by the first set of Hamilton equations for the particles.

E. The Dirac Multipliers

Once the lapse and shift functions are known, the Dirac multipliers appearing in the Dirac Hamiltonian, given in Eq.(3.48) of paper I, are determined by the following equations

$$
\begin{aligned}
\partial_\tau\, n(\tau, \vec{\sigma}) &\circeq \lambda_n(\tau, \vec{\sigma}), \\
\partial_\tau\, \bar{n}_{(a)}(\tau, \vec{\sigma}) &\circeq \lambda_{\bar{n}_{(a)}}(\tau, \vec{\sigma}).
\end{aligned}
\tag{2.13}
$$

F. The Weak ADM Energy

The weak ADM energy, given in Eqs. (6.3) and (C8) of I, becomes ($\mathcal{S}$ is given in Eq.(2.5))

$$
\begin{aligned}
\hat{E}_{ADM}|_{\theta^i=0} &= c \int d^3\sigma \Big[\mathcal{M}|_{\theta^i=0} - \frac{c^3}{16\pi\, G}\, \mathcal{S}|_{\theta^i=0} + \\
&+ \frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1} \sum_{\bar{a}} \Pi^2_{\bar{a}} + \frac{c^3}{16\pi\, G}\, \tilde{\phi} \sum_{a\neq b} \sigma^2_{(a)(b)} - \frac{c^3}{24\pi\, G}\, \tilde{\phi}\, ({}^3K)^2\Big](\tau, \vec{\sigma}),
\end{aligned}
\tag{2.14}
$$

where Eqs.(2.4) and (2.5) have to be used. As noted in paper I the gauge momentum proportional to the inertial York time ${}^3K$ (existing due to the Lorentz signature of space-time) gives rise to a negative gauge kinetic term, without analogue in ordinary gauge theories (electro-magnetism and Yang-Mills theory).

G. The Equations of Motion for the Tidal Variables

By using Eqs.(6.13), (C9), (C15), (C23) and (C7) of I, we get the expression of the momenta $\Pi_{\bar{a}}$ and then the second order equations of motion for the tidal variables $R_{\bar{a}}$. For the tidal momenta we get

$$
\begin{aligned}
\Pi_{\bar{a}}(\tau, \vec{\sigma}) &= -\frac{c^3}{8\pi\, G} \Big[\tilde{\phi} \sum_a \gamma_{\bar{a}a}\, \sigma_{(a)(a)}\Big](\tau, \vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \frac{c^3}{8\pi\, G}\, \frac{\tilde{\phi}(\tau, \vec{\sigma})}{1 + n(\tau, \vec{\sigma})} \Big[\partial_\tau\, R_{\bar{a}} + \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \\
&\Big(\Big[\gamma_{\bar{a}a}\, (2\, \partial_a\, q + \partial_a\, \Gamma^{(1)}_a)\, - \partial_a\, R_{\bar{a}}\Big]\, \bar{n}_{(a)} - \gamma_{\bar{a}a}\, \partial_a\, \bar{n}_{(a)}\Big)\Big](\tau, \vec{\sigma}),
\end{aligned}
\tag{2.15}
$$

As a consequence, for the tidal variables $R_{\bar{a}}$ we get (Eq.(2.15) is used to eliminate the dependence upon $\Pi_{\bar{a}}$ of the equation of paper I)

$$
\begin{aligned}
\partial^2_\tau\, R_{\bar{a}}(\tau, \vec{\sigma}) \stackrel{\circ}{=} &\approx \Big[\tilde{\phi}^{-1/3} \sum_a Q^{-1}_a\, \bar{n}_{(a)} \sum_{\bar{b}} (\gamma_{\bar{a}a}\, \gamma_{\bar{b}a} - \delta_{\bar{a}\bar{b}})\, \partial_a\, \partial_\tau\, R_{\bar{b}} + \\
&+ \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \Big[\Big(\gamma_{\bar{a}a}\, (2\, \partial_a q + \partial_a\, \Gamma^{(1)}_a) - \partial_a\, R_{\bar{a}}\Big)\, \bar{n}_{(a)} - \\
&- \gamma_{\bar{a}a}\, \partial_a\, \bar{n}_{(a)}\Big]\, \partial_\tau\, \Gamma^{(1)}_a - \\
&- \tilde{\phi}^{-1} \Big[\partial_\tau\, R_{\bar{a}} + \frac{2}{3}\, \tilde{\phi}^{-1/3} \sum_a Q^{-1}_a \Big(\Big[\gamma_{\bar{a}a}\, (-\partial_a\, q + \partial_a\, \Gamma^{(1)}_a) - \\
&- \partial_a\, R_{\bar{a}}\Big]\, \bar{n}_{(a)} - \gamma_{\bar{a}a}\, \partial_a\, \bar{n}_{(a)}\Big)\Big]\, \partial_\tau\, \tilde{\phi} -
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{3}\, \tilde{\phi}^{-4/3} \sum_a \gamma_{\bar{a}a}\, Q_a^{-1}\, \bar{n}_{(a)}\, \partial_a\, \partial_\tau\, \tilde{\phi} + \\
&+ \Big[\partial_\tau\, R_{\bar{a}} + \tilde{\phi}^{-1/3} \sum_a Q_a^{-1} \Big(\Big[\gamma_{\bar{a}a}\, (2\, \partial_a\, q + \partial_a\, \Gamma_b^{(1)}) - \\
&- \partial_a\, R_{\bar{a}}\Big]\, \bar{n}_{(a)} - \gamma_{\bar{a}a}\, \partial_a\, \bar{n}_{(a)}\Big)\Big]\, \frac{\partial_\tau\, n}{1+n} - \\
&- \tilde{\phi}^{-1/3} \sum_a Q_a^{-1} \Big(\Big[\gamma_{\bar{a}a}\, (2\, \partial_a\, q + \partial_a\, \Gamma_a^{(1)}) - \partial_a\, R_{\bar{a}}\Big]\, \partial_\tau\, \bar{n}_{(a)} - \\
&- \gamma_{\bar{a}a}\, \partial_a\, \partial_\tau\, \bar{n}_{(a)}\Big)\Big](\tau, \vec{\sigma}) +
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2} \Big(\tilde{\phi}^{-1}\, (1+n)\Big)(\tau, \vec{\sigma}) \int d^3\sigma_1 \Big[(1+n)(\tau, \vec{\sigma}_1)\, \frac{\delta\, \mathcal{S}(\tau, \vec{\sigma}_1)}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}|_{\theta^i=0} + n(\tau, \vec{\sigma}_1)\, \frac{\delta\, \mathcal{T}(\tau, \vec{\sigma}_1)}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}|_{\theta^i=0}\Big] + \\
&+ \Big(\tilde{\phi}^{-1}\, (1+n)\Big)(\tau, \vec{\sigma}) \Big(\frac{\tilde{\phi}^{2/3}}{1+n} \sum_a Q_a^{-1} \Big[\Big(\partial_a\, \bar{n}_{(a)} + \bar{n}_{(a)}\, (4\, \partial_a\, q - \partial_a\, \Gamma_a^{(1)} - \frac{\partial_a\, n}{1+n})\Big) \Big(\partial_\tau\, R_{\bar{a}} + \\
&+ \tilde{\phi}^{-1/3} \sum_c Q_c^{-1} \Big[\Big(\gamma_{\bar{a}c}\, (2\, \partial_c\, q + \partial_c\, \Gamma_c^{(1)})\, - \partial_c\, R_{\bar{a}}\Big)\, \bar{n}_{(c)} - \gamma_{\bar{a}c}\, \partial_c\, \bar{n}_{(c)}\Big]\Big) + \\
&+ \bar{n}_{(a)}\, \partial_a \Big(\partial_\tau\, R_{\bar{a}} + \tilde{\phi}^{-1/3} \sum_c Q_c^{-1} \Big[\Big(\gamma_{\bar{a}c}\, (2\, \partial_c\, q + \partial_c\, \Gamma_c^{(1)})\, - \partial_c\, R_{\bar{a}}\Big)\, \bar{n}_{(c)} - \gamma_{\bar{a}c}\, \partial_c\, \bar{n}_{(c)}\Big]\Big)\Big] + \\
&+ \tilde{\phi}^{2/3} \sum_{ab, a\neq b} Q_b^{-1}\, (\gamma_{\bar{a}a} - \gamma_{\bar{a}b}) \Big[\partial_b\, \bar{n}_{(a)} - (2\, \partial_b\, q + \partial_b\, \Gamma_a^{(1)})\, \bar{n}_{(a)}\Big]\, \sigma_{(a)(b)}\Big)(\tau, \vec{\sigma}) - \\
&- \frac{8\pi\, G}{c^3} \Big(\tilde{\phi}^{-1}\, (1+n)\Big)(\tau, \vec{\sigma}) \int d^3\sigma_1\, (1+n)(\tau, \vec{\sigma}_1)\, \frac{\delta\, \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}|_{\theta^i=0},
\end{aligned}
$$

$$
\int d^3\sigma_1\, \left[1 + n(\tau, \vec{\sigma}_1)\right] \frac{\delta\, \mathcal{S}(\tau, \vec{\sigma}_1)}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}|_{\theta^i=0} =
$$

$$
\begin{aligned}
&= 2 \Big(\tilde{\phi}^{1/3} \sum_a Q_a^{-2} \Big[\partial_a\, n \Big(2\, \gamma_{\bar{a}a}\, \partial_a\, q - \sum_{\bar{b}} (2\, \gamma_{\bar{a}a}\, \gamma_{\bar{b}a} - \delta_{\bar{a}\bar{b}})\, \partial_a\, R_{\bar{b}}\Big) - \\
&- (1+n) \Big(2\, \gamma_{\bar{a}a} \Big(- \partial_a^2\, q + 2\, (\partial_a\, q)^2\Big) + \\
&+ \sum_{\bar{b}} (2\, \gamma_{\bar{a}a}\, \gamma_{\bar{b}a} - \delta_{\bar{a}\bar{b}})\, (\partial_a^2\, R_{\bar{b}} + 2\, \partial_a\, q\, \partial_a\, R_{\bar{b}}) + \\
&+ \sum_{\bar{b}\bar{c}} \Big(2\, \gamma_{\bar{b}a}\, (\delta_{\bar{a}\bar{c}} - \gamma_{\bar{a}a}\, \gamma_{\bar{c}a}) - \gamma_{\bar{a}a}\, \delta_{\bar{b}\bar{c}}\Big)\, \partial_a\, R_{\bar{b}}\, \partial_a\, R_{\bar{c}}\Big)\Big]\Big)(\tau, \vec{\sigma}),
\end{aligned}
$$

$$
\int d^3\sigma_1\, n(\tau, \vec{\sigma}_1)\, \frac{\delta\, \mathcal{T}(\tau, \vec{\sigma}_1)}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}|_{\theta^i=0} = 2 \Big[\tilde{\phi}^{1/3} \sum_a \gamma_{\bar{a}a}\, Q_a^{-2} \Big(\partial_a^2\, n - 6\, \partial_a\, q\, \partial_a\, n\Big)\Big](\tau, \vec{\sigma}),
$$

$$
\begin{aligned}
&\int d^3\sigma_1 \Big(1+n(\tau,\vec{\sigma}_1)\Big)\, \frac{\delta\, \mathcal{M}(\tau,\vec{\sigma}_1)}{\delta\, R_{\bar a}(\tau,\vec{\sigma})} = \\
=&\ -\sum_i \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\, \eta_i\, \Big((1+n) \\
&\frac{\tilde{\phi}^{-2/3}\sum_a \gamma_{\bar a a}\, Q_a^{-2}\Big(\kappa_{ia}(\tau)-\frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}{\sqrt{m_i^2\, c^2+\tilde{\phi}^{-2/3}\sum_a Q_a^{-2}\Big(\kappa_{ia}(\tau)-\frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}}\Big)(\tau,\vec{\sigma}) + \\
+&\ \Big(1+n(\tau,\vec{\sigma})\Big)\Big(\tilde{\phi}^{-1/3}\Big[\frac{1}{c}\sum_{ars}\gamma_{\bar a a}\, Q_a^2\, \delta_{ra}\, \delta_{sa}\, \pi_\perp^r\, \pi_\perp^s - \frac{1}{2c}\sum_{ab}(\gamma_{\bar a a}+\gamma_{\bar a b})\, Q_a^{-2}\, Q_b^{-2}\, F_{ab}\, F_{ab} - \\
-&\ \frac{1}{c}\sum_{arsn}\gamma_{\bar a a}\, Q_a^2\, \delta_{ra}\, \delta_{sa}\Big(2\,\pi_\perp^r - \sum_m \delta^{rm}\sum_i Q_i\, \eta_i\, \partial_m\, c(\vec{\sigma},\vec{\eta}_i(\tau))\Big) \\
&\delta^{sn}\sum_j Q_j\, \eta_j\, \partial_n\, c(\vec{\sigma},\vec{\eta}_j(\tau))\Big]\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{2.16}
$$

The three integrals at the end of Eq.(2.16) were given in Eqs.(C28), (C29) and (C27) of paper I. The expression of the last integral in Eqs.(2.16) has been obtained by using the super-momentum constraints (2.7).

To get the final form of the second order equations for $R_{\bar a}$ we have to use: a) Eq.(2.8) for $\partial_\tau\, \tilde{\phi}$; b) the Hamilton equations (2.13) for the Dirac multipliers in the 3-orthogonal gauges with $\partial_\tau\, n$ and $\partial_\tau\, \bar{n}_{(r)}$ determined by the solution of Eqs. (2.12) and (2.11); c) Eq.(2.15) for $\Pi_{\bar a}$.

H. The Equations of Motion for the Particles

By using Eqs.(6.14) and (6.15) of paper I, the first half of the Hamilton equations for the particles implies

$$
\begin{aligned}
\eta_i\, \dot{\eta}_i^r(\tau)\ &\stackrel{\circ}{=}\ \eta_i\Big(\frac{\tilde{\phi}^{-2/3}\,(1+n)\, Q_r^{-2}\Big(\kappa_{ir}(\tau)-\frac{Q_i}{c}\, A_{\perp\, r}\Big)}{\sqrt{m_i^2\, c^2+\tilde{\phi}^{-2/3}\sum_c Q_c^{-2}\Big(\kappa_{ic}(\tau)-\frac{Q_i}{c}\, A_{\perp\, c}\Big)^2}} - \\
&-\ \phi^{-2}\, Q_r^{-1}\, \bar{n}_{(r)}\Big)(\tau,\vec{\eta}_i(\tau)),
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\kappa_{ir}(\tau) &= \frac{Q_i}{c}\, A_{\perp\, r}(\tau,\vec{\eta}_i(\tau)) + m_i\, c\ \ \Big[\tilde{\phi}^{2/3}\, Q_r^2\Big(\dot{\eta}_i^r(\tau)+\tilde{\phi}^{-1/3}\, Q_r^{-1}\, \bar{n}_{(r)}\Big)\ \Big(\Big(1+n\Big)^2 - \\
-&\ \tilde{\phi}^{2/3}\sum_c Q_c^2\Big(\dot{\eta}_i^c(\tau)+\tilde{\phi}^{-1/3}\, Q_c^{-1}\, \bar{n}_{(c)}\Big)^2\Big)^{-1/2}\Big](\tau,\vec{\eta}_i(\tau)).
\end{aligned}
\tag{2.17}
$$

so that the second half of the Hamilton equations becomes

$$\eta_i \frac{d}{d\tau} \qquad \Big(m_i\, c \;\; \Big(\frac{\tilde{\phi}^{2/3}\, Q_r^2\, \Big(\dot{\eta}_i^r(\tau) + \tilde{\phi}^{-1/3}\, Q_r^{-1}\, \bar{n}_{(r)}\Big)}{\sqrt{\Big(1+n\Big)^2 - \tilde{\phi}^{2/3}\sum_c Q_c^2\, \Big(\dot{\eta}_i^c(\tau) + \tilde{\phi}^{-1/3}\, Q_c^{-1}\, \bar{n}_{(c)}\Big)^2}}\Big)(\tau, \vec{\eta}_i(\tau)) \stackrel{\circ}{=}$$

$$\stackrel{\circ}{=} \Big(-\frac{\partial}{\partial\, \eta_i^r}\, \mathcal{W} + \frac{\eta_i\, Q_i}{c}\, (\dot{\eta}_i^s(\tau)\, \frac{\partial\, A_{\perp\, s}}{\partial\, \eta_i^r} - \frac{d\, A_{\perp\, r}}{d\tau}) + \eta_i\, \check{F}_{ir}\Big)(\tau, \vec{\eta}_i(\tau)),$$

$$\begin{aligned}
\mathcal{W} &= \int d^3\sigma\, \Big[(1+n)\, \mathcal{W}_{(n)} + \tilde{\phi}^{-1/3} \sum_a Q_a^{-1}\, \bar{n}_{(a)}\, \mathcal{W}_a\Big](\tau, \vec{\sigma}),\\
\mathcal{W}_{(n)}(\tau, \vec{\sigma}) &= -\frac{1}{2c}\Big[\tilde{\phi}^{-1/3} \sum_a Q_a^2\, \Big(2\, \pi_\perp^a - \delta^{am} \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^m}\Big)\\
&\qquad \delta^{an} \sum_j Q_j\, \eta_j\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial\, \sigma^n}\Big](\tau, \vec{\sigma}),\\
\mathcal{W}_r(\tau, \vec{\sigma}) &= -\frac{1}{c} \sum_s F_{rs}(\tau, \vec{\sigma})\, \delta^{sn} \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^n},
\end{aligned}$$

$$\begin{aligned}
\check{F}_{ir} &= m_i\, c\, \Big[\Big(1+n\Big)^2 - \tilde{\phi}^{2/3} \sum_c Q_c^2\, \Big(\dot{\eta}_i^c(\tau) + \tilde{\phi}^{-1/3}\, Q_c^{-1}\, \bar{n}_{(c)}\Big)^2\Big]^{-1/2}\\
&\quad \Big[-(1+n)\, \frac{\partial\, n}{\partial\, \eta_i^r} + \tilde{\phi}^{1/3} \sum_a Q_a\, \Big(\frac{\partial\, \bar{n}_{(a)}}{\partial\, \eta_i^r} -\\
&- \;(2\, \partial_r\, q + \sum_{\bar{a}} \gamma_{\bar{a}a}\, \partial_r\, R_{\bar{a}})\, \bar{n}_{(a)}\Big)\, \Big(\dot{\eta}_i^a(\tau) + \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \bar{n}_{(a)}\Big) +\\
&+ \;\tilde{\phi}^{2/3} \sum_a Q_a^2\, (2\, \partial_r\, q + \sum_{\bar{a}} \gamma_{\bar{a}a}\, \partial_r\, R_{\bar{a}})\, \Big(\dot{\eta}_i^a(\tau) + \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \bar{n}_{(a)}\Big)^2\;\Big].
\end{aligned} \tag{2.18}$$

Here $\mathcal{W}$ is the non-inertial Coulomb potential, $\check{F}_{ir}$ are inertial relativistic forces and the other terms correspond to the non-inertial Lorentz force [4].

I. The Equations of Motion for the Transverse Electro-Magnetic Field

Finally, from Eqs.(6.16) of I the Hamilton equations for the transverse electro-magnetic fields in the radiation gauge become ($P_\perp^{rs}(\vec{\sigma}) = \delta^{rs} - \sum_{uv} \delta^{ru}\, \delta^{sv}\, \frac{\partial_u\, \partial_v}{\Delta}$)

$$
\begin{aligned}
\partial_\tau\, A_{\perp\, r}(\tau, \vec{\sigma}) \,\, &\circeq \,\, \sum_{nua} \delta_{rn}\, P_{\perp}^{nu}(\vec{\sigma}) \Big[\tilde{\phi}^{-1/3}\, (1+n)\, Q_a^2\, \delta_{ua} \Big(\pi_\perp^a - \sum_m \delta^{am} \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^m} \Big) + \\
&+\,\, \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \bar{n}_{(a)}\, F_{au} \Big](\tau, \vec{\sigma}), \\
\partial_\tau\, \pi_\perp^r(\tau, \vec{\sigma}) \,\, &\circeq \,\, \sum_m P_\perp^{rm}(\vec{\sigma}) \Big(\sum_a \delta_{ma} \sum_i \eta_i\, Q_i\, \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \\
&\Big[\frac{\tilde{\phi}^{-2/3}\, (1+n)\, Q_a^{-2}\, \kappa_{ia}(\tau)}{\sqrt{m_i^2\, c^2 + \tilde{\phi}^{-2/3}\, \sum_b Q_b^{-2}\, \Big(\kappa_{ib}(\tau) - \frac{Q_i}{c}\, A_{\perp\, b}\Big)^2}} - \\
&-\,\, \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \bar{n}_{(a)} \Big](\tau, \vec{\eta}_i(\tau)) - \\
&\\
-\,\, &\Big[2\, \tilde{\phi}^{-1/3}\, (1+n) \sum_{ab} Q_a^{-2}\, Q_b^{-2}\, \delta_{ma} \Big(\partial_b\, F_{ab} - \Big[2\, \partial_b\, q + 2\, \partial_b\, (\Gamma_a^{(1)} + \Gamma_b^{(1)}) \Big]\, F_{ab} \Big) + \\
+\,\, &2\, \tilde{\phi}^{-1/3} \sum_{ab} Q_a^{-2}\, Q_b^{-2}\, \delta_{ma}\, \partial_b\, n\, F_{ab} - \\
-\,\, &\tilde{\phi}^{-1/3} \sum_a \bar{n}_{(a)}\, Q_a^{-1} \Big(\partial_a\, \pi_\perp^m - \Big[2\, \partial_a\, q + \partial_a\, \Gamma_a^{(1)} \Big]\, \pi_\perp^m + \\
+\,\, &\delta_{ma} \sum_n \Big[2\, \partial_n\, q + \partial_n\, \Gamma_a^{(1)} \Big]\, \pi_\perp^n + \\
+\,\, &\sum_i \eta_i\, Q_i \Big[\Big(2\, \partial_a\, q + \partial_a\, \Gamma_a^{(1)} \Big) \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau)))}{\partial\, \sigma^m} - \frac{\partial^2\, c(\vec{\sigma}, \vec{\eta}_i(\tau)))}{\partial\, \sigma^m\, \partial\, \sigma^a} - \\
-\,\, &\delta_{ma} \sum_n \Big(\Big[2\, \partial_n\, q + \partial_n\, \Gamma_a^{(1)} \Big] \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau)))}{\partial\, \sigma^n} - \frac{\partial^2\, c(\vec{\sigma}, \vec{\eta}_i(\tau)))}{\partial\, \sigma^n\, \partial\, \sigma^n} \Big) \Big] \Big) + \\
+\,\, &\tilde{\phi}^{-1/3} \sum_a Q_a^{-1} \sum_i \eta_i\, Q_i \Big(\partial_a\, \bar{n}_{(a)}\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^m} - \delta_{ma} \sum_n \partial_n\, \bar{n}_{(a)}\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^n} \Big)\, \Big](\tau, \vec{\sigma}) \Big).
\end{aligned}
\tag{2.19}
$$

J. The Weak ADM Poincare' Generators

While the weak ADM energy $\hat{P}^\tau_{ADM} = \frac{1}{c}\, \hat{E}_{ADM}$ is given in Eq.(2.14), Eqs.(2.22) and (3.47) of paper I give the following expressions in the 3-orthogonal gauges for the other weak ADM Poincare' generators (the last term in the boosts was added in Ref.[19])

$$
\begin{aligned}
\hat{P}^r_{ADM} &= \int d^3\sigma \Big[{}^3g^{rs}\,\mathcal{M}_s - 2\,{}^3\Gamma^r_{su}(\tau,\vec{\sigma})\,{}^3\Pi^{su}\Big](\tau,\vec{\sigma}) = \\
&= 2\int d^3\sigma\,\Big\{\,\tilde{\phi}^{-2/3}\sum_{\bar{b}}\,Q_r^{-2}\Big(2\,\gamma_{\bar{b}r}\,\partial_r\,q + \sum_{\bar{a}}(\gamma_{\bar{b}r}\gamma_{\bar{a}r} - \frac{1}{2}\,\delta_{\bar{a}\bar{b}})\,\partial_r R_{\bar{a}}\Big)\,\Pi_{\bar{b}} - \\
&- \frac{c^3}{12\pi\,G}\,\tilde{\phi}^{1/3}\,Q_r^{-2}\Big(4\,\partial_r\,q + \partial_r\,\Gamma_r^{(1)}\Big)\,{}^3K + \\
&+ \frac{c^3}{8\pi G}\tilde{\phi}^{1/3}\sum_d\,Q_r^{-1}Q_d^{-1}\Big(2\,\partial_d\,q + \partial_d\,\Gamma_r^{(1)}\Big)\,\sigma_{(r)(d)} + \frac{1}{2}\tilde{\phi}^{-2/3}\,Q_r^{-2}\,\mathcal{M}_r\Big\} \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}^{rs}_{ADM} &= \int d^3\sigma\,\Big[-2(\sigma^r\,{}^3\Gamma^s_{uv} - \sigma^s\,{}^3\Gamma^r_{uv})\,{}^3\Pi^{uv} + (\sigma^r\,{}^3g^{su} - \sigma^s\,{}^3g^{ru})\,\mathcal{M}_u\Big](\tau,\vec{\sigma}) = \\
&= 2\int d^3\sigma\,\Big\{\,\sigma^r\,\Big[\,\tilde{\phi}^{-2/3}\sum_{\bar{b}}\,Q_s^{-2}\Big(2\,\gamma_{\bar{b}s}\,\partial_s\,q + \sum_{\bar{a}}(\gamma_{\bar{b}s}\gamma_{\bar{a}s} - (1/2)\delta_{\bar{a}\bar{b}})\,\partial_s R_{\bar{a}}\Big)\,\Pi_{\bar{b}} - \\
&- \frac{c^3}{12\pi\,G}\,\tilde{\phi}^{1/3}\,Q_s^{-2}\,\Big(4\,\partial_s\,q + \partial_s\,\Gamma_s^{(1)}\Big)\,{}^3K + \\
&+ \frac{c^3}{8\pi G}\tilde{\phi}^{1/3}\sum_d\,Q_s^{-1}Q_d^{-1}\Big(2\,\partial_d\,q + \partial_d\,\Gamma_s^{(1)}\Big)\,\sigma_{(s)(d)} + \frac{1}{2}\tilde{\phi}^{-2/3}\,Q_s^{-2}\,\mathcal{M}_s\,\Big] - \\
&- \sigma^s\,\Big[\,\tilde{\phi}^{-2/3}\sum_{\bar{b}}\,Q_r^{-2}\Big(2\,\gamma_{\bar{b}r}\,\partial_r\,q + \sum_{\bar{a}}(\gamma_{\bar{b}r}\gamma_{\bar{a}r} - (1/2)\delta_{\bar{a}\bar{b}})\,\partial_r R_{\bar{a}}\Big)\,\Pi_{\bar{b}} - \\
&- \frac{c^3}{12\pi\,G}\,\tilde{\phi}^{1/3}\,Q_r^{-2}\,\Big(4\,\partial_r\,q + \partial_r\,\Gamma_r^{(1)}\Big)\,{}^3K + \\
&+ \frac{c^3}{8\pi G}\tilde{\phi}^{1/3}\sum_d\,Q_r^{-1}Q_d^{-1}\Big(2\,\partial_d\,q + \partial_d\,\Gamma_r^{(1)}\Big)\,\sigma_{(r)(d)} + \frac{1}{2}\tilde{\phi}^{-2/3}\,Q_r^{-2}\,\mathcal{M}_r\,\Big]\,\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}^{\tau r}_{ADM} &= -\hat{J}^{r\tau}_{ADM} = \int d^3\sigma\,\Big(\sigma^r\,\Big[\frac{c^3}{16\pi\,G}\,\sqrt{\gamma}\;{}^3g^{ns}\,({}^3\Gamma^u_{nv}\,{}^3\Gamma^v_{su} - {}^3\Gamma^u_{ns}\,{}^3\Gamma^v_{vu}) - \\
&- \frac{8\pi\,G}{c^3\sqrt{\gamma}}\,{}^3G_{nsuv}\,{}^3\Pi^{ns}\,{}^3\Pi^{uv} - \mathcal{M}\Big] + \\
&+ \frac{c^3}{16\pi\,G}\,\delta^r_u\,({}^3g_{vs} - \delta_{vs})\,\partial_n\,\Big[\sqrt{\gamma}\,({}^3g^{ns}\,{}^3g^{uv} - {}^3g^{nu}\,{}^3g^{sv})\Big]\Big)(\tau,\vec{\sigma}) =
\end{aligned}
$$

$$
\begin{aligned}
= & \int d^3\sigma \Big\{\sigma^r \Big[\frac{c^3}{16\pi G}\,\mathcal{S} - \mathcal{M} - \frac{4\pi G}{c^3}\tilde{\phi}^{-1}\sum_{\bar{b}} \Pi^2_{\bar{b}} - \\
- & \tilde{\phi}\,\frac{c^3}{16\pi G}\Big(\sum_{a\neq b}\sigma^2_{(a)(b)} - \frac{2}{3}\,({}^3K)^2\Big)\Big] - \\
- & \frac{c^3}{16\pi G}\tilde{\phi}^{-1/3}\,Q_r^{-2}\sum_s(\tilde{\phi}^{2/3} - Q_s^{-2})\Big[\delta_{rs}\,\partial_s\,(\Gamma_r^{(1)} + \Gamma_s^{(1)} + q) - \\
- & \partial_r\,(\Gamma_r^{(1)} + \Gamma_s^{(1)} + q)\Big]\Big\}(\tau,\vec{\sigma}) \approx 0.
\end{aligned}
\tag{2.20}
$$

As discussed in Section IIE of paper I, $\hat{J}^{\tau r}_{ADM} \approx 0$ are the gauge-fixings for the rest-frame constraints $\hat{P}^r_{ADM} \approx 0$ eliminating the internal 3-center of mass in the 3-spaces Σ_τ, which are non-inertial rest frames of the 3-universe.

K. Dimensions

In checking the validity of the previous formulas it is useful to remember the dimensions of the relevant quantities:

$[\tau = ct] = [x^\mu] = [\vec{\sigma}] = [\vec{\eta}_i] = [l]$, $[\vec{\kappa}_i] = [m_i c] = [P^\mu] = [E/c] = [m\,l\,t^{-1}]$, $[S] = [\hbar] = [J^{AB}] = [m\,l^2\,t^{-1}]$, $[T^{AB}] = [\mathcal{M}] = [\mathcal{M}_r] = [\mathcal{H}] = [\mathcal{H}_{(a)}] = [m\,l^{-2}\,t^{-1}]$, $[{}^4g] = [{}^3g] = [n] = [n_{(a)}] = [{}^3e_{(a)r}] = [\dot{\vec{\eta}}_i] = [\theta_i] = [\tilde{\phi}] = [0]$, $[{}^3\pi^r_{(a)}] = [{}^3\tilde{\Pi}^{rs}] = [\Pi_{\bar{a}}] = [\pi_{\tilde{\phi}}] = [m\,l^{-1}\,t^{-1}]$, $[{}^3\omega_{r(a)}] = [{}^3K_{rs}] = [{}^3K] = [\sigma_{(a)(b)}] = [l^{-1}]$, $[{}^3R] = [{}^3\Omega_{rs(a)}] = [l^{-2}]$, $[Q_i] = [m^{1/2}\,l^{3/2}\,t^{-1}]$, $[A_{\perp\,r}] = [Q\,l^{-1}] = [m^{1/2}\,l^{1/2}\,t^{-1}]$, $[\frac{Q_i}{c}\,A_{\perp r}] = [m\,l\,t^{-1}]$, $[\pi^r_\perp] = [E^r_\perp] = [B_r] = [l^{-1}\,A_{\perp\,r}] = [m^{1/2}\,l^{-1/2}\,t^{-1}]$, $[G = 6.7\,10^{-8}\,cm^3\,s^{-2}\,g^{-1}] = [m^{-1}\,l^3\,t^{-2}]$, $[G/c^3 = 2.5\,10^{-39}\,sec/g] = [m^{-1}\,t]$, $[G/c^2 = 7.421\,10^{-29}\,cm/g] = [m^{-1}\,l]$.

III. THE WEAK FIELD APPROXIMATION AND THE LINEARIZATION

The standard decomposition used for the weak field approximation in the harmonic gauges is

$$
{}^4g_{\mu\nu} \;=\; {}^4\eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}|, |\partial_\alpha\, h_{\mu\nu}|, |\partial_\alpha\, \partial_\beta\, h_{\mu\nu}| << 1,
\tag{3.1}
$$

where ${}^4\eta_{\mu\nu}$ is the flat metric in an inertial frame of the background Minkowski space-time. This is equivalent to take a 3+1 splitting of our space-time with an inertial foliation, having Euclidean instantaneous 3-spaces, against the equivalence principle and against the fact (explicitly shown in paper I) that each solution of Einstein's equations has an associated dynamically selected preferred 3+1 splitting.

In this Section we shall define a linearization of Hamilton-Dirac equations in the (non-harmonic) 3-orthogonal Schwinger time gauges (2.1) using as background the asymptotic Minkowski 4-metric existing in our asymptotically Minkowskian space-times. Actually we look for the following decomposition to be done by using radar 4-coordinates adapted to an admissible (see Ref.[4]) 3+1 splitting of space-time

$$
{}^4g_{AB}(\tau, \sigma^r) \;=\; {}^4g_{(1)AB}(\tau, \sigma^r) + O(\zeta^2) \;\rightarrow {}^4\eta_{AB(asym)} \; at\, spatial\, infinity,
$$

$$
\begin{aligned}
&{}^4g_{(1)AB}(\tau, \sigma^r) = {}^4\eta_{AB(asym)} + {}^4h_{(1)AB}(\tau, \sigma^r),\\
&{}^4h_{(1)AB}(\tau, \sigma^r) = O(\zeta) \rightarrow 0 \; at\, spatial\, infinity,
\end{aligned}
\tag{3.2}
$$

where $\zeta << 1$ is a small a-dimensional parameter, the small perturbation ${}^4h_{(1)AB}$ has no intrinsic meaning in the bulk and ${}^3g_{(1)rs}(\tau, \sigma^r) = -\epsilon\, {}^4g_{(1)rs}(\tau, \sigma^r)$ is the positive-definite 3-metric on the instantaneous 3-space Σ_τ. In our case the instantaneous 3-spaces will deviated from flat Euclidean 3-spaces by curvature effects of order $O(\zeta)$, in accord with the equivalence principle.

We must make an assumption on the variables $R_{\bar a}$, $\partial_\tau\, R_{\bar a}$, $\partial_\tau^2\, R_{\bar a}$, ϕ, n, $\bar n_{(a)}$, $\sigma_{(a)(b)}|_{a\not= b}$, $\Pi_{\bar a}$ (or $\sigma_{(a)(a)}$), $\pi_{\tilde \phi} = \frac{c^3}{12\pi\, G}\, {}^3K$, $\dot \eta^r_i$, κ_{ir}, $A_{\perp\, r}$, $\pi^r_{\perp}$ such that the coupled equations (2.8) and (2.9) (contracted Bianchi identities), (2.11) (shift determination), (2.12) (lapse determination with free ${}^3K$), (2.15) (expression of $\Pi_{\bar a}$), (2.16) (equations for $\partial_\tau^2\, R_{\bar a}$), (2.17) (expression of κ_{ir}), (2.18) (equations for η^r_i), (2.19) (Hamilton equations for $A_{\perp\, r}$ and $\pi^r_{\perp}$), all have the two members consistent (of the same order).

Let us remember [14] that to avoid coordinate singularities we must always have $N(\tau, \vec \sigma) = 1 + n(\tau, \vec \sigma) > 0$ (3-spaces at different times do not intersect each other), $\epsilon\, {}^4g_{\tau\tau}(\tau, \vec \sigma) > 0$ (no rotating disk pathology) and ${}^3g_{rs}(\tau, \vec \sigma)$ with three distinct positive eigenvalues.

A. A Consistent Hamiltonian Linearization

Let us see which assumptions are needed to get Eq.(3.2) in the 3-orthogonal Schwinger time gauges (2.1).

The first assumption is that on each instantaneous 3-space Σ_τ we have the following limitation of the a-dimensional configurational tidal variables $R_{\bar a}$ in the York canonical basis

$$
\begin{aligned}
&|R_{\bar a}(\tau,\vec\sigma) = R_{(1)\bar a}(\tau,\vec\sigma)| = O(\zeta) << 1,\\
&|\partial_u\, R_{\bar a}(\tau,\vec\sigma)| \sim \frac{1}{L} O(\zeta),\qquad |\partial_u\,\partial_v\, R_{\bar a}(\tau,\vec\sigma)| \sim \frac{1}{L^2} O(\zeta),\\
&|\partial_\tau\, R_{\bar a}| = \frac{1}{L}\, O(\zeta),\qquad |\partial^2_\tau\, R_{\bar a}| = \frac{1}{L^2}\, O(\zeta),\qquad |\partial_\tau\,\partial_u\, R_{\bar a}| = \frac{1}{L^2}\, O(\zeta),\\
&\Rightarrow\;\; Q_a(\tau,\vec\sigma) = e^{\sum_{\bar a}\,\gamma_{\bar a a}\, R_{\bar a}(\tau,\vec\sigma)} = 1 + \Gamma_a^{(1)}(\tau,\vec\sigma) + O(\zeta^2),\\
&\qquad \Gamma_a^{(1)} = \sum_{\bar a}\gamma_{\bar a a}\, R_{\bar a} = O(\zeta),\qquad \sum_a \Gamma_a^{(1)} = 0,\qquad R_{\bar a} = \sum_a \gamma_{\bar a a}\,\Gamma_a^{(1)},
\end{aligned}
\tag{3.3}
$$

where L is a *big enough characteristic length interpretable as the reduced wavelength $\lambda/2\pi$ of the resulting GW's*. Therefore the tidal variables $R_{\bar a}$ are slowly varying over the length L and times L/c. This also implies that the Riemann tensor ${}^4R_{ABCD}$, the Ricci tensor ${}^4R_{AB}$ and the scalar 4-curvature ${}^4R$ behave as $\frac{1}{L^2}\, O(\zeta)$. Also the intrinsic 3-curvature scalar of the instantaneous 3-spaces Σ_τ, given in Eqs.(2.5), is of order $\frac{1}{L^2}\, O(\zeta)$. To simplify the notation we use $R_{\bar a}$ for $R_{(1)\bar a}$ in the rest of the paper.

As a consequence of the behavior of the Riemann tensor, the mean radius of curvature ${}^4\mathcal{R}$ of space-time is of order ${}^4\mathcal{R}^{-2} \approx \frac{1}{L^2}\, O(\zeta)$. Therefore we get that the *requirements of the weak field approximation are satisfied*:

i) $\mathcal{A} = O(\zeta)$, if $\mathcal{A} \sim R_{\bar a}$ is the amplitude of the GW;

ii) $\left(\frac{L}{\mathcal{R}}\right)^2 = O(\zeta)$, namely $L \approx \frac{\lambda}{2\pi} << {}^4\mathcal{R}$.

As a first attempt let us put $\phi = \tilde\phi^{1/6} = 1 + \phi_{(o)} + \phi_{(1)} + O(\zeta^2)$, $n = n_{(o)} + n_{(1)} + O(\zeta^2)$ and $\bar n_{(a)} = \bar n_{(o)(a)} + \bar n_{(1)(a)} + O(\zeta^2)$, with $\phi_{(o)}, n_{(o)}, \bar n_{(o)(a)} = O(1)$, $\phi_{(1)}, n_{(1)}, \bar n_{(1)(a)} \sim O(\zeta)$, and with similar expansions for the other variables like $\sigma_{(a)(b)}$, ${}^3K$, ... However, this implies [5]

[5] For $|\phi_{(1)}| < 1$ we have $\phi^n = \tilde\phi^{n/6} = (1+\phi_{(o)})^n + n\,(1+\phi_{(o)})^{n-1}\,\phi_{(1)} \;\mapsto_{\phi_{(o)}=0}\; 1 + n\,\phi_{(1)} + O(\zeta^2)$, $\phi^{-n} = \tilde\phi^{-n/6} = (1+\phi_{(o)})^{-n} - n\,(1+\phi_{(o)})^{-n-1}\,\phi_{(1)} \;\mapsto_{\phi_{(o)}=0}\; 1 - n\,\phi_{(1)} + O(\zeta^2)$, $\partial_r\, q = \phi^{-1}\,\partial_r\,\phi = \frac{1}{6}\,\tilde\phi^{-1}\,\partial_r\,\tilde\phi = (1+\phi_{(o)})^{-1}\,\partial_r\,\phi_{(o)} + (1+\phi_{(o)})^{-2}\Big[(1+\phi_{(o)})\,\partial_r\,\phi_{(1)} - \phi_{(1)}\,\partial_r\,\phi_{(o)}\Big] \;\mapsto_{\phi_{(o)}=0}\; \partial_r\,\phi_{(1)} + O(\zeta^2)$.

$$
\begin{aligned}
-\,\epsilon\, {}^4g_{rs} &= {}^3g_{rs} = \phi^4 \Big(1 + 2 \sum_{\bar a} \gamma_{\bar ar}\, R_{\bar a}\Big)\, \delta_{rs} + O(\zeta^2) = \\
&= \Big((1+\phi_{(o)})^4 + 4\,(1+\phi_{(o)})^3\, \phi_{(1)} + 2\,(1+\phi_{(o)})^4 \sum_{\bar a} \gamma_{\bar ar}\, R_{\bar a}\Big)\, \delta_{rs} + O(\zeta^2) = \\
&= (1+\phi_{(o)})^4\, \delta_{rs} + O(\zeta), \\
\epsilon\, {}^4g_{\tau\tau} &= (1+n)^2 - \sum_a \bar n^2_{(a)} = (1+n_{(o)})^2 - \sum_a \bar n^2_{(o)(a)} + O(\zeta), \\
-\epsilon\, {}^4g_{\tau r} &= \phi^2\, Q_r\, \bar n_{(r)} = \Big((1+\phi_{(o)})^2 + 2\,(1+\phi_{(o)})\, \phi_{(1)} + \\
&+\; (1+\phi_{(o)})^2 \sum_{\bar a} \gamma_{\bar ar}\, R_{\bar a}\Big)\, \bar n_{(r)} + O(\zeta^2) = (1+\phi_{(o)})^2\, \bar n_{(o)(r)} + O(\zeta),
\end{aligned}
\tag{3.4}
$$

and the equations for $\phi_{(o)}$, $n_{(o)}$, $\bar n_{(o)(a)}$ turn out to be not linear.

Therefore we must assume $\phi_{(o)} = n_{(o)} = \bar n_{(o)(a)} = 0$. In this way Eq.(3.2) can be implemented in the following way [6]

$$
\begin{aligned}
\phi &= 1 + \phi_{(1)} + O(\zeta^2), \qquad \tilde\phi = 1 + 6\, \phi_{(1)} + O(\zeta^2), \\
N &= 1 + n = 1 + n_{(1)} + O(\zeta^2), \qquad \bar n_{(a)} = \bar n_{(1)(a)} + O(\zeta^2), \\
&\Downarrow \qquad {}^4g_{(1)AB} = {}^4\eta_{AB(asym)} + {}^4h_{(1)AB}, \\
{}^4h_{(1)\tau\tau} &= 2\, \epsilon\, n_{(1)} = O(\zeta), \\
{}^4h_{(1)\tau r} &= -\epsilon\, \bar n_{(1)(r)} = O(\zeta), \\
{}^4h_{(1)rs} &= -2\, \epsilon\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})\, \delta_{rs} = O(\zeta), \qquad \Gamma_r^{(1)} = \sum_{\bar a} \gamma_{\bar ar}\, R_{\bar a}, \quad \sum_r \Gamma_r^{(1)} = 0., \\
& h_{(1)} = \epsilon\, {}^4\eta^{AB}\, {}^4h_{(1)AB} = 2\,(n_{(1)} - 6\, \phi_{(1)}) = O(\zeta),
\end{aligned}
\tag{3.5}
$$

while the triads and cotriads become ${}^3\bar e^r_{(1)(a)} = \delta^r_a\, (1 - \Gamma_r^{(1)} - 2\, \phi_{(1)}) + O(\zeta^2)$ and ${}^3\bar e_{(1)(a)r} = \delta_{ra}\, (1 + \Gamma_r^{(1)} + 2\, \phi_{(1)}) + O(\zeta^2)$, respectively. Therefore we have ${}^4g_{\tau\tau} = \epsilon\, [1 + 2\, n_{(1)}] + O(\zeta^2)$, ${}^4g_{\tau r} = -\epsilon\, \bar n_{(1)(r)} + O(\zeta^2)$, ${}^4g_{rs} = -\epsilon\, {}^3g_{rs} = -\epsilon\, \delta_{rs}\, [1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})] + O(\zeta^2)$, $\tilde\phi = \phi^6 = \sqrt{det\, {}^3g_{rs}} = 1 + 6\, \phi_{(1)} + O(\zeta^2)$.

With these assumptions Eq.(2.3) implies

[6] The "trace reversed" perturbation is ${}^4\bar h_{(1)AB} = {}^4h_{(1)AB} - \frac{1}{2}\, {}^4\eta_{AB}\, \epsilon\, h_{(1)}$, $\bar h_{(1)} = -h_{(1)}$, ${}^4\bar h_{(1)\tau\tau} = \epsilon\, (6\, \phi_{(1)} - n_{(1)})$, ${}^4\bar h_{(1)\tau r} = -\epsilon\, \bar n_{(1)(r)}$, ${}^4\bar h_{(1)rs} = -\epsilon\, [2\, (\Gamma_r^{(1)} + 5\, \phi_{(1)}) - n_{(1)}]\, \delta_{rs}$.

$$\frac{8\pi\, G}{c^3}\,\Pi_{\bar a}(\tau,\vec\sigma) \;=\; \frac{8\pi\, G}{c^3}\,\Pi_{(1)\bar a}(\tau,\vec\sigma) = \frac{1}{L}\,O(\zeta)\stackrel{\circ}{=}\Big[\partial_\tau\, R_{\bar a} - \sum_a \gamma_{\bar a a}\,\partial_a\,\bar n_{(1)(a)}\Big](\tau,\vec\sigma) + \frac{1}{L}\,O(\zeta^2),$$

$$\sigma_{(a)(a)} = \sigma_{(1)(a)(a)} = -\frac{8\pi\, G}{c^3}\sum_{\bar a}\gamma_{\bar a a}\,\Pi_{(1)\bar a} + \frac{1}{L}\,O(\zeta^2). \tag{3.6}$$

Let us remark that everywhere $\Pi_{(1)\bar a}$ appears in the combination $\frac{G}{c^3}\,\Pi_{(1)\bar a} = \frac{1}{L}\,O(\zeta)$, which behaves like $\partial_\tau\, R_{\bar a}$, i.e. it varies slowly over L.

Finally the super-momentum constraints (2.7), Eqs.(2.3) and dimensional arguments require

$$\begin{aligned}
&\sigma_{(a)(b)}|_{a\not= b} = \sigma_{(1)(a)(b)}|_{a\not= b} = \frac{1}{L}\,O(\zeta),\\
&\Rightarrow\quad \frac{8\pi\, G}{c^3}\,\pi_i^{(\theta)} = \frac{1}{L}\,O(\zeta^2) = \sum_{a\not= b}(\Gamma_a^{(1)} - \Gamma_b^{(1)})\,\epsilon_{iab}\,\sigma_{(1)(a)(b)} + \frac{1}{L}\,O(\zeta^3),
\end{aligned}$$

$$\begin{aligned}
&{}^3K = \frac{12\pi\, G}{c^3}\,\pi_{\tilde\phi} = {}^3K_{(1)} = \frac{12\pi\, G}{c^3}\,\pi_{(1)\tilde\phi} = \frac{1}{L}\,O(\zeta),\\
&\Downarrow
\end{aligned}$$

$$\begin{aligned}
{}^3K_{rs} \;&=\; {}^3K_{(1)rs} = \frac{1}{L}\,O(\zeta) = \\
&=\; (1-\delta_{rs})\,\sigma_{(1)(r)(s)} + \delta_{rs}\,\Big[\frac{1}{3}\,{}^3K_{(1)} - \partial_\tau\,\Gamma_r^{(1)} + \sum_a(\delta_{ra} - \frac{1}{3})\,\partial_a\,\bar n_{(1)(a)}\Big] + \frac{1}{L}\,O(\zeta^2).
\end{aligned} \tag{3.7}$$

These equations imply that once we have found a solution $\sigma_{(1)(a)(b)}|_{a\not= b}(\tau,\vec\sigma)$ of the linearization of the super-momentum constraints (2.7), then we can put $\pi_i^{(\theta)}(\tau,\vec\sigma)\approx 0$ in the York canonical basis, which becomes adapted to 13 of the 14 constraints after the linearization.

Let us remark that the triad momenta and the standard ADM momenta, given after Eq.(2.22) of paper I, have the following weak field limit: $\pi^r_{(a)} = \delta_{ra}\,(\pi_{(1)\tilde\phi} + \sum_{\bar b}\gamma_{\bar b a}\,\Pi_{(1)\bar b}) - \frac{c^3}{8\pi\, G}\,(1-\delta_{rs})\,\sigma_{(1)(r)(s)} + \frac{1}{L}\,O(\zeta^2)$, ${}^3\Pi^{rs} = \frac{1}{4}\,({}^3\bar e^r_{(a)}\,{}^3\bar\pi^s_{(a)} + {}^3\bar e^s_{(a)}\,{}^3\bar\pi^r_{(a)}) = \frac{1}{4}\,({}^3\pi^r_{(s)} + \pi^s_{(r)}) + \frac{1}{L}\,O(\zeta^2) = -\frac{1}{2}\,\delta_{rs}\,(\pi_{(1)\tilde\phi} + \sum_{\bar b}\gamma_{\bar b r}\,\Pi_{(1)\bar b}) + \frac{c^3}{16\pi\, G}\,(1-\delta_{rs})\,\sigma_{(1)(r)(s)} + \frac{1}{L}\,O(\zeta^2)$.

Let us now consider our matter, i.e. positive-energy scalar particles and the transverse electro-magnetic field in the radiation gauge.

For the particles we have $\eta^r_i = O(1)$ and $\dot\eta^r_i = O(1)$ (since $\tau = c\,t$, in the non-relativistic limit we have $\dot{\vec\eta}_i = \vec v_i/c = O(1)\;\rightarrow_{c\rightarrow\infty}\;0$).

However, without further restrictions on the masses, the momenta and the electro-magnetic field Eqs.(2.4) would imply $\mathcal{M} = \mathcal{M}_{(o)} + \mathcal{M}_{(1)} + \frac{mc}{L^3}\,O(\zeta^2)$, with $\mathcal{M}_{(o)} = O(1)$,

$\mathcal{M}_{(1)} = \frac{mc}{L^3}\, O(\zeta)$, and $\mathcal{M}_r = \mathcal{M}_{(o)\, r} = \frac{mc}{L^3}\, O(1)$ ($\mathcal{M}$ and $\mathcal{M}_r$ are densities; m is a typical particle mass). But then Eqs.(2.6) and (2.7) for the super-Hamiltonian and super-momentum constraints would not be consistent. For instance Eq.(2.6), whose unknown is $\phi = 1 + \phi_{(1)} + O(\zeta^2)$, would be $\mathcal{M}_{(o)}(\tau, \vec{\sigma}) + F_{(1)}[\phi_{(1)}, \mathcal{M}_{(1)}, ...](\tau, \vec{\sigma}) + \frac{mc}{L^3}\, O(\zeta^2) \approx 0$ with $F_{(1)} = \frac{mc}{L^3}\, O(\zeta)$.

To get a consistent approximation we must introduce a *ultraviolet cutoff* M on the masses and momenta of the particles and on the electro-magnetic field so that

$$
\begin{aligned}
\mathcal{M}(\tau, \vec{\sigma}) \;&=\; \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) + \mathcal{R}_{(2)}(\tau, \vec{\sigma}),\\
&\qquad m_i = M\, O(\zeta), \qquad \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) = M c\, O(\zeta), \qquad \int d^3\sigma\, \mathcal{R}_{(2)}(\tau, \vec{\sigma}) = M c\, O(\zeta^2),\\
\mathcal{M}_r(\tau, \vec{\sigma}) \;&=\; \mathcal{M}_{(1) r}(\tau, \vec{\sigma}), \qquad \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1) r}(\tau, \vec{\sigma}) = M c\, O(\zeta).
\end{aligned}
\tag{3.8}
$$

Here M is a finite mass defining the ultraviolet cutoff: $M\, c^2$ gives an estimate of the weak ADM energy of the 3-universe contained in the instantaneous 3-spaces Σ_τ, because it can be assumed to be of the order of the mass Casimir of the asymptotic ADM Poincare' group. The associated length scale is the gravitational radius $R_M = 2 M\, \frac{G}{c^2} \approx 10^{-29}\, M$ [7].

Therefore the description of particles in our approximation will be reliable only if their masses and momenta are less of $M c\, O(\zeta)$ and at distances r from the particles satisfying $r > R_M$ (that is at each instant we must enclose each particle in a sphere of radius R_M and our approximation is not valid inside these spheres). This will be clear in the next Section, where we will obtain an equation like $\triangle\, \phi_{(1)}(\tau, \vec{\sigma}) = \frac{1}{L^2}\, O(\zeta) \sim \frac{G}{c^3}\, \check{\mathcal{M}}_{(1)}(\tau, \vec{\sigma}) + ...$ implying $\phi_{(1)}(\tau, \vec{\sigma}) = O(\zeta) \sim \frac{G}{c^3}\, \frac{m_i\, c}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} + ... \sim \frac{R_M}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}\, O(\zeta)$ namely $|\vec{\sigma} - \vec{\eta}_i(\tau)| >> R_M$. Therefore our results in the weak field approximation can be trusted till a distance $d >> R_M$ from the particles.

From Eq.(2.4) we get for the mass density and mass current density

---

[7] The Earth mass $M_{Earth} = 5.98\, 10^{28}\, g$ gives rise to a gravitational radius $R_{Earth} = \frac{2\, G\, M_{Earth}}{c^2} = 0.888\, cm$. By comparison the Compton wavelength of an electron is $\frac{\hbar}{m_e\, c} = 3.861592\, 10^{-11} cm$, the classical electron radius is $\frac{e^2}{m_e\, c^2} = 2.81794\, 10^{-13}\, cm$ and the Planck length is $L_P = \sqrt{\frac{\hbar\, G}{c^3}} = 1.616\, 10^{-33} cm$.

$$\mathcal{M}^{(UV)}(\tau,\vec{\sigma}) \; = \; \mathcal{M}^{(UV)}_{(1)}(\tau,\vec{\sigma}) + \mathcal{M}^{(UV)}_{(2)}(\tau,\vec{\sigma}) + \mathcal{R}_{(3)}(\tau,\vec{\sigma}), \qquad \int d^3\sigma\, \mathcal{R}(\tau,\vec{\sigma}) = Mc\, O(\zeta^3)$$

$$\begin{aligned}
\mathcal{M}^{(UV)}_{(1)}(\tau,\vec{\sigma}) \; &= \; \sum_i \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\, \eta_i \sqrt{m_i^2\, c^2 + \Big(\vec{\kappa}_i(\tau) - \frac{Q_i}{c}\, \vec{A}_\perp\Big)^2}(\tau,\vec{\sigma}) + \\
&+ \; \frac{1}{2c}\Big(\Big[\sum_a \Big((\pi^a_\perp)^2 - \Big(2\,\pi^a_\perp - \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\, \sigma^a}\Big) \\
&\sum_j Q_j\, \eta_j\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_j(\tau))}{\partial\, \sigma^a}\Big) + \frac{1}{2}\sum_{ab} F^2_{ab}\Big]\Big)(\tau,\vec{\sigma}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(UV)}_{(2)}(\tau,\vec{\sigma}) \; &= \; \sum_i \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\, \eta_i \Big(\\
&\frac{-2\,\phi_{(1)}\Big(\vec{\kappa}_i(\tau) - \frac{Q_i}{c}\, \vec{A}_\perp\Big)^2 - \sum_a \Gamma^{(1)}_a \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp a}\Big)^2}{\sqrt{m_i^2\, c^2 + \Big(\vec{\kappa}_i(\tau) - \frac{Q_i}{c}\, \vec{A}_\perp\Big)^2}}\Big)(\tau,\vec{\sigma}) - \\
&- \; \frac{1}{2c}\Big(2\sum_a \Big[\phi_{(1)} - \Gamma^{(1)}_a\Big]\Big[(\pi^a_\perp)^2 - \Big(2\,\pi^a_\perp - \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\, \sigma^a}\Big) \\
&\sum_j Q_j\, \eta_j\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_j(\tau))}{\partial\, \sigma^a}\Big] + \sum_{ab} \Big[\phi_{(1)} + \Gamma^{(1)}_a + \Gamma^{(1)}_b\Big]\, F^2_{ab}\Big)(\tau,\vec{\sigma}), \\
&\int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)}(\tau,\vec{\sigma}) = Mc\, O(\zeta), \qquad \int d^3\sigma\, \mathcal{M}^{(UV)}_{(2)}(\tau,\vec{\sigma}) = Mc\, O(\zeta^2),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\mathcal{M}_r(\tau,\vec{\sigma}) \; &= \; \mathcal{M}^{(UV)}_{(1)r}(\tau,\vec{\sigma}) = \sum_{i=1}^N \eta_i \Big(\kappa_{ir}(\tau) - \frac{Q_i}{c}\, A_{\perp r}(\tau,\vec{\sigma})\Big)\, \delta^3(\vec{\sigma},\vec{\eta}_i(\tau)) - \\
&- \; \frac{1}{c}\sum_s F_{rs}(\tau,\vec{\sigma})\Big(\pi^s_\perp(\tau,\vec{\sigma}) - \sum_n \delta^{sn}\sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\, \sigma^n}\Big), \\
&\int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)r}(\tau,\vec{\sigma}) = Mc\, O(\zeta).
\end{aligned} \tag{3.10}$$

Therefore for the particles and the transverse electro-magnetic field the validity of the weak field approximation requires

$$
\begin{aligned}
&\vec{\eta}_i(\tau) = O(1), \qquad \frac{\vec{\kappa}_i(\tau)}{m_i c} = O(1), \qquad \frac{\vec{\kappa}_i(\tau)}{M c} = O(\zeta), \qquad \frac{m_i}{M} \leq O(\zeta),\\
&A_{\perp r}(\tau, \vec{\sigma}), \pi^r_{\perp}(\tau, \vec{\sigma}) = O(1), \qquad with \quad \frac{Q_i}{c}\, A_{\perp r}(\tau, \vec{\eta}_i(\tau)) = Mc\, O(\zeta),\\
&\int d^3\sigma\, [\frac{1}{c}\, \pi^r_{\perp}(\tau, \vec{\sigma})]^2,\; \frac{1}{c}\, F^2_{rs}(\tau, \vec{\sigma}),\; [\frac{1}{c}\, F_{rs}\, \pi^s_{\perp}](\tau, \vec{\sigma}) = Mc\, O(\zeta).
\end{aligned} \tag{3.11}
$$

Moreover the boundary conditions at spatial infinity and the local intensities for the transverse electro-magnetic field must be such that the integral conditions in Eqs.(3.8) hold. The last line of Eqs.(3.11) agrees with Eq.(3.8) only if the radiation part of the transverse electro-magnetic field is concentrated in small volumes $V = V_o\, O(\zeta)$ with a sufficiently rapid decay outside them. The restriction $\frac{Q_i}{c}\, A_{\perp r}(\tau, \vec{\eta}_i(\tau)) = Mc\, O(\zeta)$, dictated by Eq.(2.17), implies a bound on the value of the electric charges of the Lienard-Wiechert transverse potential evaluated in Ref.[15] in special relativity: since this potential has the form $\frac{Q_i}{c}\, A_{\perp r\, LW}(\tau, \vec{\sigma}) = \sum_{j\neq i} \frac{Q_i\, Q_j}{c\, 4\pi\, |\vec{\sigma} - \vec{\eta}_j(\tau)|}\, F_{jr}$ with $F_{jr} = O(1)$, it turns out that for distances $|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)| > R_M$ the restriction implies that the product $e_i\, e_j$ of the electric charges semiclassically simulated by the Grassmann variables $Q_i\, Q_j$ must satisfy $e_i\, e_j < R_M\, M\, c^2$.

Our results will be equivalent to a re-summation of the PN expansions valid for small rest masses still having relativistic velocities ($\frac{\tilde{\vec{\kappa}}_i^2}{m_i^2\, c^2} = O(1)$, $\frac{\vec{v}_i}{c} = O(1)$).

Let us remark that in this way the energy-momentum tensor T^{AB} (its expression derives from Eq.(3.11) of paper I after having expressed the metric components in the York canonical basis, after having expressed the electro-magnetic field in the radiation gauge as said after Eq.(3.35) of paper I and after having done the weak field approximation) has the following behavior

$$
\begin{aligned}
T^{\tau\tau} &= \mathcal{M}^{(UV)}_{(1)} + \mathcal{R}^{\tau\tau}_{(2)},\\
T^{\tau r} &= \mathcal{M}^{(UV)}_{(1)\, r} + \mathcal{R}^{\tau r}_{(2)},\\
T^{rs} &= \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i)\, \eta_i\, \frac{(\kappa_{ir} - \frac{Q_i}{c}\, A_{\perp\, r})\, (\kappa_{is} - \frac{Q_i}{c}\, A_{\perp\, s})}{\sqrt{m_i^2\, c^2 + \sum_a (\kappa_{ia} - \frac{Q_i}{c}\, A_{\perp\, a})^2}} +\\
&+ \frac{1}{c}\Big[- \Big(\pi^r_{\perp} - \sum_i \eta_i\, Q_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^r}\Big) \Big(\pi^s_{\perp} - \sum_j \eta_j\, Q_j\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial\, \sigma^s}\Big) +\\
&+ \frac{1}{2}\, \delta^{rs}\, \Big(\sum_a (\pi^a_{\perp} - \sum_i \eta_i\, Q_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^a})^2 - \frac{1}{2} \sum_{ab} F^2_{ab}\Big) + \sum_a F_{ra}\, F_{sa}\Big] + \mathcal{R}^{rs}_{(2)} =\\
&= T^{rs}_{(1)} + \mathcal{R}^{rs}_{(2)},
\end{aligned}
$$

$$\sum_r T^{rr}_{(1)} = \sum_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))\, \eta_i\, \frac{\sum_r \Big(\kappa_{ir} - \frac{Q_i}{c}\, A_{\perp r}\Big)^2}{\sqrt{m_i^2\, c^2 + \sum_a (\kappa_{ia} - \frac{Q_i}{c}\, A_{\perp\, a})^2}} +$$
$$+\ \frac{1}{2c}\Big[\sum_a (\pi^a_\perp)^2 + \frac{1}{2}\sum_{ab} F^2_{ab} -$$
$$-\ \sum_{ai} \eta_i\, Q_i\, \Big(2\, \pi^a_\perp - \sum_{j\neq i} \eta_j\, Q_j\, \partial_a\, c(\vec{\sigma}, \vec{\eta}_j(\tau))\Big)\, \partial_a\, c(\vec{\sigma}, \vec{\eta}_i(\tau))\Big],$$

$$\sum_r \gamma_{\bar{a}r}\, T^{rr}_{(1)} = \sum_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))\, \eta_i\, \frac{\sum_r \gamma_{\bar{a}r}\, \Big(\kappa_{ir} - \frac{Q_i}{c}\, A_{\perp r}\Big)^2}{\sqrt{m_i^2\, c^2 + \sum_a (\kappa_{ia} - \frac{Q_i}{c}\, A_{\perp\, a})^2}} -$$
$$-\ \frac{1}{c}\sum_r \gamma_{\bar{a}r}\, \Big[(\pi^r_\perp)^2 - \sum_a F^2_{ra} -$$
$$-\ \sum_i \eta_i\, Q_i\, \Big(2\, \pi^r_\perp - \sum_{j\neq i} \eta_j\, Q_j\, \partial_r\, c(\vec{\sigma}, \vec{\eta}_j(\tau))\Big)\, \partial_r\, c(\vec{\sigma}, \vec{\eta}_i(\tau))\Big],$$

$$\mathcal{M}^{(UV)}_{(2)} = -\sum_a (\Gamma^{(1)}_a + 2\, \phi_{(1)})\, T^{aa}_{(1)} +$$
$$+\ \frac{1}{c}\Big[\sum_a 3\, \Gamma^{(1)}_a\, \Big(\pi^a_\perp - \sum_i \eta_i\, Q_i\, \frac{\partial\, c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^a}\Big)^2 -$$
$$-\ \sum_{ab} (\phi_{(1)} + \frac{3}{2}\, \Gamma^{(1)}_a)\, F^2_{ab}\Big]. \tag{3.12}$$

Since, as said in Subsection IIE of paper I, we have $\nabla_A\, T^{AB}(\tau, \vec{\sigma}) \circeq 0$ from the Bianchi identities and since ${}^4g_{AB} = {}^4\eta_{AB(asym)} + O(\zeta)$, we must have $\partial_A\, T^{AB}_{(1)}(\tau, \vec{\sigma}) \circeq 0 + \partial_A\, \mathcal{R}^{AB}_{(2)}$. At the lowest order this implies

$$\partial_\tau\, \mathcal{M}^{(UV)}_{(1)} + \partial_r\, \mathcal{M}^{(UV)}_{(1)r} = 0 + \partial_A\, \mathcal{R}^{A\tau}_{(2)},$$
$$\partial_\tau\, \mathcal{M}^{(UV)}_{(1)r} + \partial_s\, T^{rs}_{(1)} = 0 + \partial_A\, \mathcal{R}^{Ar}_{(2)}, \tag{3.13}$$

as in inertial frames in Minkowski space-time. The equation $\partial_A\, T^{AB}_{(1)}(\tau, \vec{\sigma}) \circeq 0 + \partial_A\, \mathcal{R}^{AB}_{(2)}$ implies $\partial_A\, \Big(T^{AB}_{(1)}(\tau, \vec{\sigma})\, \sigma^C - T^{AC}_{(1)}(\tau, \vec{\sigma})\, \sigma^B\Big) \circeq 0 + \partial_A\, \mathcal{R}^{ABC}_{(2)}$ (angular momentum conservation).

Finally let us consider Einstein's equations in radar 4-coordinates, i.e. ${}^4R_{AB} - \frac{1}{2}\, {}^4g_{AB}\, {}^4R \circeq \frac{8\pi\, G}{c^3}\, T_{AB}$. Since we have ${}^4R_{AB} = \frac{1}{L^2}\, O(\zeta)$ from Eqs.(3.3) and $\frac{8\pi\, G}{c^3} \int d^3\sigma\, T_{(1)AB}(\tau, \vec{\sigma}) \approx R_M\, O(\zeta)$ from Eqs. (3.8) and (3.12), we get from Einstein's equations the following local estimate of the order of $T_{(1)AB}(\tau, \vec{\sigma})$

$$T_{(1)AB}(\tau, \vec{\sigma}) \approx \frac{Mc}{R_M\, L^2}\, O(\zeta). \tag{3.14}$$

Therefore the support $V = V_o\, O(\zeta)$ of the radiation part of the electro-magnetic field, defined after Eq.(3.11), must have $V_o \sim R_M\, L^2$ with $L \geq R_M$.

In conclusion, since the weak field linearized solution can be trusted only at distances $d >> R_M$ from the particles, the GW's described by our linearization must have a wavelength satisfying $\lambda \approx L > d >> R_M$ (with the weak field approximation we have $\lambda << {}^4\mathcal{R}$ without the slow motion assumption).

If all the particles are contained in a compact set of radius l_c (the source), the frequency $\nu = \frac{c}{\lambda}$ of the emitted GW's will be of the order of the typical frequency ω_s of the motion inside the source, where the typical velocities are of the order $v \approx \omega_s\, l_c$. As a consequence we get $\nu = \frac{c}{\lambda} \approx \omega_s \approx v/l_c$ or $\lambda \approx \frac{c}{v}\, l_c >> R_M$, so that we get $\frac{v}{c} \approx \frac{l_c}{\lambda} << \frac{l_c}{R_M}$ and $l_c >> R_M$ if $\frac{v}{c} = O(1)$.

If the velocities of the particles become non-relativistic, i.e. in the slow motion regime with $v << c$ (for binary systems with total mass m and held together by weak gravitational forces we have also $\frac{v}{c} \approx \sqrt{\frac{R_m}{l_c}} << 1$), we have $\lambda >> l_c$ and we can have $l_c \approx R_M$.

B. The Linearization Interpreted as the First Term of a Hamiltonian Post-Minkowskian Expansion in the Non-Harmonic 3-Orthogonal Gauges

In our class of asymptotically flat space-times near Minkowski space-time the above linearization can be interpreted as the first term in a HPM expansion with a UV cutoff on the matter. This is due to the fact that most of the canonical variables in the York basis parametrize deviations from Minkowski space-time with the Cartesian 4-coordinates of an inertial frame, which vanish if $G \to 0$ (for $G = 0$ we get $R_{\bar{a}} = n = \bar{n}_{(a)} = \Pi_{\bar{a}} = 0$, $\phi = 1$ in the Minkowski rest-frame instant form, where we have also ${}^3K = \sigma_{(a)(b)} = 0$).

As a consequence, by using Eqs.(2.3) we can write the following HPM expansions (here we do not use $R_{\bar{a}}$ to mean $R_{(1)\bar{a}}$ as is done in the weak field approximation)

$$
\begin{aligned}
R_{\bar a} &= \sum_{n=1}^{\infty} G^n\, R_{[n]\,\bar a}, \qquad R_{(1)\bar a} = G\, R_{[1]\bar a},\\
n &= \sum_{n=1}^{\infty} G^n\, n_{[n]}, \qquad n_{(1)} = G\, n_{[1]},\\
\bar n_{(a)} &= \sum_{n=1}^{\infty} G^n\, \bar n_{[n]\,(a)}, \qquad \bar n_{(1)(a)} = G\, \bar n_{[1](a)},\\
\phi &= e^q = \tilde\phi^{1/6} = 1 + \sum_{n=1}^{\infty} G^n\, \phi_{[n]}, \qquad \phi_{(1)} = G\, \phi_{[1]},\\
{}^3K &= \frac{12\pi}{c^3}\, G\, \pi_{\tilde\phi} = \sum_{n=1}^{\infty} G^n\, {}^3K_{[n]}, \qquad {}^3K_{(1)} = G\, {}^3K_{[1]},\\
&\qquad \pi_{\tilde\phi} = \frac{c^3}{12\pi}\Big({}^3K_{[1]} + \sum_{n=2}^{\infty} G^{n-1}\, {}^3K_{[n]}\Big),\\
\sigma_{(a)(b)}|_{a\not= b} &= \frac{8\pi}{c^3}\, \phi^{-6}\, \sum_i \frac{\epsilon_{abi}\, G\, \pi_i^{(\theta)}}{Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1}} = \sum_{n=1}^{\infty} G^n\, \sigma_{[n]\,(a)(b)}|_{a\not= b}, \qquad \sigma_{(1)\,(a)(b)}|_{a\not= b} = G\, \sigma_{[1]\,(a)(b)}|_{a\not= b}.
\end{aligned}
\tag{3.15}
$$

Moreover Eq.(3.6) implies $G\, \Pi_{\bar a} = \sum_{n=1}^{\infty} G^n\, \Pi_{[n]\,\bar a}$, so that from Eq.(2.3) we get $\sigma_{(a)(a)} - \frac{8\pi}{c^3}\, \phi^{-6}\, \sum_{\bar a} \gamma_{\bar aa}\, G\, \Pi_{\bar a} = \sum_{n=1}^{\infty} G^n\, \sigma_{[n](a)(a)}$. Finally, since we have $Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1} \rightarrow_{G \rightarrow 0} 2\, \sum_{\bar a} (\gamma_{\bar aa} - \gamma_{\bar ab})\, G\, R_{[1]\bar a}$, from $G\, \sigma_{[1](a)(b)}|_{a\not= b} = \frac{4\pi}{c^3}\, \sum_i \frac{\epsilon_{abi}\, \pi_i^{(\theta)}}{\sum_{\bar a} (\gamma_{\bar aa} - \gamma_{\bar ab})\, R_{[1]\bar a}}$ we also get $\pi_i^{(\theta)} = \sum_{n=1}^{\infty} G^n\, \pi^{(\theta)}_{[n]i}$.

The study of HPM at the second order will be done in a future paper.

C. The HPM Linearization of the Gauge Fixings for Harmonic Gauges

In Eqs. (5.3) and (5.4) of paper I we expressed the Hamiltonian version of the gauge-fixing constraints $\chi^\tau(\tau, \vec \sigma) \approx 0$ and $\chi^r(\tau, \vec \sigma) \approx 0$ selecting the family of 4-harmonic gauges in the York canonical basis.

If we eliminate the gauge fixing $\theta^i(\tau, \vec \sigma) \approx 0$ identifying the family of 3-orthogonal gauges and we consider the angles (O(3) canonical coordinates of first kind) $\theta^i(\tau, \vec \sigma)$ as small quantities $\theta^i_{(1)}(\tau, \vec \sigma) = O(\zeta)$ (so that we have $V_{sa}(\theta^i) = \delta_{sa} + \epsilon_{sai}\, \theta^i_{(1)} + O(\zeta^2)$), we can extend our HPM linearization to arbitrary gauges. With some calculatins it can be checked that the first half of Eqs. (3.7), regarding the super-momentum constraints, are still valid with $\theta^i = \theta^i_{(1)} + O(\zeta^2) \not= 0$.

Then the linearization of the harmonic gauge fixing $\chi^\tau(\tau, \vec \sigma) \approx 0$ (see the second half of Eqs. (5.3) of paper I) becomes

$$
\partial_\tau\, n_{(1)}(\tau, \vec \sigma) \approx - \Big(\sum_r \partial_r\, \bar n_{(1)(r)} + {}^3K_{(1)}\Big)(\tau, \vec \sigma).
\tag{3.16}
$$

Instead the linearization of the harmonic gauge fixings $\chi^r(\tau, \vec{\sigma}) \approx 0$ of Eqs. (5.4) of paper I becomes

$$\partial_\tau\, \bar{n}_{(1)(r)}(\tau, \vec{\sigma}) \approx -\Big(\partial_r\, n_{(1)} + 2\, \partial_r\, (\phi_{(1)} - \Gamma_r^{(1)})\Big)(\tau, \vec{\sigma}). \tag{3.17}$$

Both the equations do not depend on $\theta_{(1)}(\tau, \vec{\sigma})$. As said in paper I all the gauge fixings for the gauge variables $\theta_{(1)}(\tau, \vec{\sigma})$ and ${}^3K_{(1)}(\tau, \vec{\sigma})$ compatible with these equations identify 4-harmonic gauges.

The previous two equations imply the following wave equations for the lapse and shift functions

$$\begin{aligned}
&\Box\, n_{(1)}(\tau, \vec{\sigma}) \approx \Big(2 \sum_r \partial_r^2\, (\phi_{(1)} - \Gamma_r^{(1)})\Big)(\tau, \vec{\sigma}),\\
&\Box\, \bar{n}_{(1)(r)}(\tau, \vec{\sigma}) - \partial_r \sum_{s \neq r} \bar{n}_{(1)(s)}(\tau, \vec{\sigma}) \approx\\
&\approx\ \Big(2\, \partial_r\, \partial_\tau\, (\Gamma_r^{(1)} - \phi_{(1)}) + \partial_r\, {}^3K_{(1)}\Big)(\tau, \vec{\sigma}).
\end{aligned} \tag{3.18}$$

The solution of these hyperbolic equations requires initial data at $\tau \rightarrow -\infty$. Instead in the family of 3-orthogonal gauges we have elliptic equations on a fixed 3-space Σ_τ requiring data only on it.

If we denote $n_{(1)}^{(HH)}$ and $\bar{n}_{(1)(r)}^{(HH)}$ the retarded solutions of Eqs.(3.18), the 4-metric ${}^4g_{(1)AB}^{(HH)}(\bar{\tau}, \vec{\bar{\sigma}})$ in harmonic radar 4-coordinates $(\bar{\tau}, \vec{\bar{\sigma}})$ will have the form ${}^4g_{(1)\tau\tau}^{(HH)} = \epsilon\Big(1 + 2\, n_{(1)}^{(HH)}\Big)$, ${}^4g_{(1)\tau r}^{(HH)} = -\epsilon\, \bar{n}_{(1)(r)}^{(HH)}$, ${}^4g_{(1)rs}^{(HH)} = -\epsilon\, \Big(\delta_{rs} + A_{(1)rs}^{(HH)}\Big)$ with $A_{(1)rs}^{(HH)}$ depending on which harmonic gauge one chooses. The connection to the 4-metric ${}^4g_{(1)AB}(\tau, \vec{\sigma})$ in the family of 3-orthogonal gauges is by means of a 4-coordinate transformation $\tau = \bar{\tau} + a_{(1)}(\bar{\tau}, \vec{\bar{\sigma}})$, $\sigma^r = \bar{\sigma}^r + b_{(1)}^r(\bar{\tau}, \vec{\bar{\sigma}})$ implying ${}^4g_{(1)AB}^{(HH)}(\bar{\tau}, \vec{\bar{\sigma}}) = \frac{\partial\, \sigma^C}{\partial\, \bar{\sigma}^A}\, \frac{\partial\, \sigma^D}{\partial\, \bar{\sigma}^B}\, {}^4g_{(1)CD}(\tau, \vec{\sigma})$. As a consequence one gets

$$\begin{aligned}
&\frac{\partial\, a_{(1)}(\bar{\tau}, \vec{\bar{\sigma}})}{\partial\, \bar{\tau}} = \Big(n_{(1)}^{(HH)} - n_{(1)}\Big)(\bar{\tau}, \vec{\bar{\sigma}}),\\
&\frac{\partial\, b_{(1)}^r(\bar{\tau}, \vec{\bar{\sigma}})}{\partial\, \bar{\tau}} = \Big(\bar{n}_{(1)(r)}^{(HH)} - \bar{n}_{(1)(r)}\Big)(\bar{\tau}, \vec{\bar{\sigma}}),\\
&\frac{\partial\, b_{(1)}^r(\bar{\tau}, \vec{\bar{\sigma}})}{\partial\, \bar{\sigma}^s} + \frac{\partial\, b_{(1)}^s(\bar{\tau}, \vec{\bar{\sigma}})}{\partial\, \bar{\sigma}^r} = \Big(2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})\, \delta_{rs} - A_{(1)rs}^{(HH)}\Big)(\bar{\tau}, \vec{\bar{\sigma}}).
\end{aligned} \tag{3.19}$$

IV. THE SOLUTION OF THE LINEARIZED EQUATIONS FOR $\tilde{\phi}$, $1+n$, $\pi_i^{(\theta)}$, $\bar{n}_{(a)}$ AND THE LINEARIZED ADM GENERATORS

In this Section we find the linearization of the super-Hamiltonian and super-momentum constraints and then of the equations determining the lapse and shift functions of our family of 3-orthogonal Schwinger time gauges. The solutions of these linearized equations allow to express $\tilde{\phi}_{(1)}$, $1+n_{(1)}$, $\sigma_{(1)(a)(b)}|_{a\neq b}$ (i.e. $\pi_i^{(\theta)}$), $\bar{n}_{(1)(a)}$ in terms of matter and of the tidal variables $\Gamma_r^{(1)} = \sum_{\bar{a}} \gamma_{\bar{a}r} R_{\bar{a}}$. In the next Section we will see that the linearized equations for the tidal variables depend only on matter, so that at the end the previous solutions will depend only on the matter. While the equations of elliptic type solved in this Section will determine the instantaneous inertial dependence on the matter of the 4-metric (like the Coulomb potential in the radiation gauge in the case of the electro-magnetic field), the wave equations for the tidal variables in the next Section will determine the retarded dependence on matter of the 4-metric. We will also see that at this order the contracted Bianchi identities are identically satisfied. Then we will evaluate the asymptotic ADM Poincare' generators till the second order.

A. The Super-Hamiltonian Constraint and the Lapse Function

Let us first consider the determination of $\phi = \tilde{\phi}^{1/6} = 1 + \phi_{(1)} + O(\zeta^2)$ by using the super-Hamiltonian constraint (2.6) and of the lapse function by means of Eq. (2.12).

1. *The Equation for* $\phi_{(1)}$

Since the linearized Laplace-Beltrami operator and the scalar 3-curvature in the family of 3-orthogonal gauges, see Eq.(2.5), have the expressions ($\triangle = \sum_a \partial_a^2$ is the flat asymptotic Laplacian)

$$\begin{aligned} \hat{\triangle}|_{\theta^i=0} &= \triangle + O(\zeta), \qquad \hat{\triangle}|_{\theta^i=0}\, \phi = \triangle\, \phi_{(1)} + O(\zeta^2), \\ {}^3\hat{R}|_{\theta^i=0} &= 2 \sum_a \partial_a^2\, \Gamma_a^{(1)} + O(\zeta^2), \end{aligned} \tag{4.1}$$

the super-Hamiltonian constraint given in Eq.(2.6) gives rise to the following linearized elliptic equation for $\phi_{(1)}$

$$\triangle\, \phi_{(1)}(\tau, \vec{\sigma}) \approx -\frac{2\pi\, G}{c^3}\, \mathcal{M}_{(1)}^{(UV)}(\tau, \vec{\sigma}) + \frac{1}{4} \sum_a \partial_a^2\, \Gamma_a^{(1)}(\tau, \vec{\sigma}) + O(\zeta^2). \tag{4.2}$$

2. *The Equation for the Lapse Function* $1 + n_{(1)}$

In the family of 3-orthogonal gauges Eqs.(3.7) imply ${}^3K = {}^3K_{(1)} = \frac{1}{L}\, O(\zeta) \approx F_{(1)}$ with $F_{(1)}(\tau, \vec{\sigma})$ arbitrary numerical function of the same order.

To find the linearization of Eq.(2.12) for the lapse function we need the following result

$$\int d^3\sigma_1 \Big(1 + n(\tau, \vec{\sigma}_1)\Big) \frac{\delta\, \mathcal{M}(\tau, \vec{\sigma}_1)}{\delta\, \phi(\tau, \vec{\sigma})} =$$

$$= -2 \sum_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau))\, \eta_i \Big(\frac{\sum_a \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}{\sqrt{m_i^2\, c^2 + \sum_a \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}} \Big)(\tau, \vec{\sigma}) -$$

$$-\frac{1}{c} \Big[\sum_a (\pi^a_\perp)^2 + \frac{1}{2} \sum_{ab} F^2_{ab} - \sum_a \Big(2\, \pi^a_\perp - \sum_{k \neq j} \eta_k\, Q_k\, \partial_a\, c(\vec{\sigma}, \vec{\eta}_k(\tau)) \Big)$$

$$\sum_j \eta_j\, Q_j\, \partial_a\, c(\vec{\sigma}, \vec{\eta}_j(\tau)) \Big](\tau, \vec{\sigma}) - 2 \sum_a \mathcal{R}^{aa}_{(2)} =$$

$$= -2 \sum_a T^{aa}_{(1)} - 2 \sum_a \mathcal{R}^{aa}_{(2)}, \qquad from\ Eq.(3.12), \tag{4.3}$$

where the following approximation was used

$$\frac{1}{\sqrt{m_i^2\, c^2 + \tilde{\phi}^{-2/3} \sum_a Q_a^{-2} \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}}(\tau, \vec{\eta}_i(\tau)) =$$

$$= \Big(\frac{1}{\sqrt{m_i^2\, c^2 + \sum_a \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}} \Big[1 +$$

$$+ \frac{\sum_a (\Gamma^{(1)}_a + 2\, \phi_{(1)}) \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2}{m_i^2\, c^2 + \sum_a \Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp\, a}\Big)^2} \Big] \Big)(\tau, \vec{\eta}_i(\tau)) + \frac{1}{Mc}\, O(\zeta). \tag{4.4}$$

As a consequence, by using Eqs. (4.3) and (3.12), Eq.(2.12) becomes the following linearized elliptic equation for the lapse function

$$\begin{aligned} \triangle\, n_{(1)}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} -\partial_\tau\, {}^3K_{(1)}(\tau, \vec{\sigma}) + \frac{1}{2} \sum_a \partial_a^2\, (\Gamma^{(1)}_a - 4\, \phi_{(1)})(\tau, \vec{\sigma}) + \frac{4\pi\, G}{c^3} \sum_a T^{aa}_{(1)}(\tau, \vec{\sigma}) = \\ &\stackrel{(4.2)}{=} -\partial_\tau\, {}^3K_{(1)}(\tau, \vec{\sigma}) + \frac{4\pi\, G}{c^3} \Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)} \Big)(\tau, \vec{\sigma}). \end{aligned} \tag{4.5}$$

3. The Solutions for $\phi_{(1)}$ and $n_{(1)}$

The solutions, vanishing at spatial infinity, of Eqs. (4.2) and (4.5) for $\phi_{(1)}$ and $n_{(1)}$ are ($\mathcal{M}^{(UV)}_{(1)}$ is given in Eq.(3.9))

$$
\begin{aligned}
\phi_{(1)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[-\frac{2\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} + \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big](\tau,\vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} \frac{G}{2c^3}\Big[\sum_i \eta_i\,\frac{\sqrt{m_i^2\,c^2 + \sum_c\Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\,A_{\perp\,c}(\tau,\vec{\eta}_i(\tau))\Big)^2}}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} + \\
&+ \frac{1}{2\,c}\int \frac{d^3\sigma_1}{|\vec{\sigma} - \vec{\sigma}_1|}\Big(\Big[\sum_a\Big((\pi^a_\perp)^2 - \Big(2\,\pi^a_\perp - \sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\,\sigma^a}\Big) \\
&\sum_j Q_j\,\eta_j\,\frac{\partial\,c(\vec{\sigma},\vec{\eta}_j(\tau))}{\partial\,\sigma^a}\Big) + \frac{1}{2}\sum_{ab} F^2_{ab}\Big]\Big)(\tau,\vec{\sigma}_1)\Big] - \\
&- \frac{1}{16\pi}\int d^3\sigma_1\,\frac{\sum_a \partial^2_{1a}\,\Gamma_a^{(1)}(\tau,\vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|}, \qquad \partial_r\,q = \partial_r\,\phi_{(1)} + O(\zeta^2), \\
{}^4g_{(1)rs}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} -\epsilon\,\delta_{rs}\,[1 + 2\,(\Gamma_r^{(1)} + 2\,\phi_{(1)})(\tau,\vec{\sigma})],
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
n_{(1)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - \frac{1}{\triangle}\,\partial_\tau\,{}^3K_{(1)}\Big](\tau,\vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=} -\frac{G}{c^3}\Big[\sum_i \eta_i\,\frac{\sqrt{m_i^2\,c^2 + \sum_c\Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\,A_{\perp\,c}(\tau,\vec{\eta}_i(\tau))\Big)^2}}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \\
&\Big(1 + \frac{\sum_c\Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\,A_{\perp\,c}(\tau,\vec{\eta}_i(\tau))\Big)^2}{m_i^2\,c^2 + \sum_c\Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\,A_{\perp\,c}(\tau,\vec{\eta}_i(\tau))\Big)^2}\Big) + \\
&+ \frac{1}{c}\int \frac{d^3\sigma_1}{|\vec{\sigma} - \vec{\sigma}_1|}\Big[\sum_a\Big((\pi^a_\perp)^2 - \Big(2\,\pi^a_\perp - \sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\,\sigma^a}\Big) \\
&\sum_j Q_j\,\eta_j\,\frac{\partial\,c(\vec{\sigma},\vec{\eta}_j(\tau))}{\partial\,\sigma^a}\Big) + \frac{1}{2}\sum_{ab} F^2_{ab}\Big](\tau,\vec{\sigma}_1)\Big] + \\
&+ \int d^3\sigma_1\,\frac{\partial_\tau\,{}^3K(\tau,\vec{\sigma}_1)}{4\pi\,|\vec{\sigma} - \vec{\sigma}_1|}, \\
{}^4g_{(1)\tau\tau}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \epsilon\,[1 + 2\,n_{(1)}(\tau,\vec{\sigma})].
\end{aligned}
\tag{4.7}
$$

While $\phi_{(1)}$ depends upon the tidal variables, the lapse function of these 3-orthogonal gauges is independent from them. However, while the volume 3-element $\phi_{(1)}$ of the 3-space is independent from the inertial gauge variable ${}^3K_{(1)}$, the lapse function, connecting nearby instantaneous 3-space with different local York times, depends upon $\partial_\tau\,{}^3\mathcal{K}_{(1)} = \partial_\tau\,\frac{1}{\triangle}\,{}^3K_{(1)}$. At this order the spatial 4-metric ${}^4g_{(1)rs}$ depends upon $2\,\phi_{(1)} + \Gamma_r^{(1)}$: the first term describes

the instantaneous inertial part of the gravitational field in the 3-orthogonal gauges, while the tidal term the retarded one. Instead the component ${}^4g_{(1)\tau\tau}$, relevant for local proper time, depends only upon the instantaneous inertial part of the gravitational field and upon the inertial gauge variable $\partial_\tau\, {}^3\mathcal{K}_{(1)}$.

B. The Super-Momentum Constraints and the Shift Functions

Let us now consider the determination of $\sigma_{(1)(a)(b)}|_{a\neq b}$, by means of the super-momentum constraints (2.7) and the determination of the shift functions from Eqs. (2.11). As said after Eq.(3.7), such a solution $\sigma_{(1)(a)(b)}|_{a\neq b}$ implies $\pi_i^{(\theta)} = \frac{c^3}{8\pi G} \sum_{a\neq b} \epsilon_{iab}\, (\Gamma_a^{(1)} - \Gamma_b^{(1)})\, \sigma_{(1)(a)(b)} \approx 0$ in the York canonical basis of the linearized theory.

1. The Equations for $\sigma_{(1)(a)(b)}|_{a\neq b}$

Eqs.(2.7) for $\sigma_{(1)(a)(b)}|_{a\neq b}$ takes the following linearized form ($\mathcal{M}^{(UV)}_{(1)a}$ is given in Eq.(3.10))

$$\begin{aligned}
\sum_{b\neq a} \partial_b\, \sigma_{(1)(a)(b)}(\tau,\vec{\sigma}) \;&\approx\; \frac{2}{3}\, \partial_a\, {}^3K_{(1)}(\tau,\vec{\sigma}) + \frac{8\pi\, G}{c^3}\, \Big(\mathcal{M}^{(UV)}_{(1)a} + \sum_{\bar{a}} \gamma_{\bar{a}a}\, \partial_a\, \Pi_{\bar{a}}\Big)(\tau,\vec{\sigma}) = \\
&= \Big(\partial_\tau\, \partial_a\, \Gamma_a^{(1)} + \frac{1}{3}\, \partial_a\, (\sum_c \partial_c\, \bar{n}_{(1)(c)}) - \partial_a^2\, \bar{n}_{(1)(a)} + \\
&+\; \frac{2}{3}\, \partial_a\, {}^3K_{(1)} + \frac{8\pi\, G}{c^3}\, \mathcal{M}^{(UV)}_{(1)a}\Big)(\tau,\vec{\sigma}),
\end{aligned} \tag{4.8}$$

where we have used Eq.(3.6) for $\Pi_{\bar{a}}$ and $\sum_{\bar{a}} \gamma_{\bar{a}a}\, \gamma_{\bar{a}b} = \delta_{ab} - \frac{1}{3}$ (see before Eq.(2.3)).

2. The Equations for the Shift Functions $\bar{n}_{(1)(r)}$

Eqs.(2.11) with $a \neq b$ for the shift functions gives rise to the following linearized equations

$$\Big(\partial_b\, \bar{n}_{(1)(a)}(\tau,\vec{\sigma}) + \partial_a\, \bar{n}_{(1)(b)}(\tau,\vec{\sigma})\Big)|_{a\neq b} \,\circeq\, 2\, \sigma_{(1)(a)(b)}|_{a\neq b}(\tau,\vec{\sigma}), \qquad a \neq b. \tag{4.9}$$

3. The Solutions for $\bar{n}_{(1)(a)}$ and $\sigma_{(1)(a)(b)}|_{a\neq b}$

By applying the operator ∂_b to Eqs.(4.9) and by summing over b, we get

$$\sum_{b\neq a} \Big[\partial_b^2\, \bar{n}_{(1)(a)} + \partial_a\, (\partial_b\, \bar{n}_{(1)(b)})\Big](\tau,\vec{\sigma}) \,\circeq\, 2 \sum_{b\neq a} \partial_b\, \sigma_{(1)(a)(b)}(\tau,\vec{\sigma}). \tag{4.10}$$

By putting Eqs.(4.8) for $\sigma_{(1)(a)(b)}|_{a\neq b}$ into Eq.(4.10), we get an equation containing only the shift functions

$$\sum_{b\not= a}\, \Big[\partial_b^2\, \bar n_{(1)(a)} + \partial_a\, (\partial_b\, \bar n_{(1)(b)})\Big](\tau ,\vec \sigma) \,{\buildrel \circ \over =}\,$$
$$ {\buildrel \circ \over =}\, \Big[{4\over 3}\, \partial_a\, {}^3K_{(1)} + {{16\pi\, G}\over {c^3}}\, {\cal M}^{(UV)}_{(1)a} - 2\, \partial_a^2\, \bar n_{(1)(a)} +$$
$$ + {2\over 3}\, \partial_a\, (\sum_c\, \partial_c\, \bar n_{(1)(c)}) + 2\, \partial_{\tau}\, \partial_a\, \Gamma_a^{(1)}\Big](\tau ,\vec \sigma). \tag{4.11}$$

Eqs.(4.11) can be rewritten in the form (no more containing the condition $b \not= a$)

$$\Big[\triangle\, \bar n_{(1)(a)} + {1\over 3}\, \partial_a\, (\sum_b\, \partial_b\, \bar n_{(1)(b)})\Big](\tau ,\vec \sigma) \,{\buildrel \circ \over =}\,$$
$$ {\buildrel \circ \over =}\, \Big[{4\over 3}\, \partial_a\, {}^3K_{(1)} + 2\, \partial_{\tau}\, \partial_a\, \Gamma_a^{(1)} + {{16\pi\, G}\over {c^3}}\, {\cal M}^{(UV)}_{(1)a}\Big](\tau ,\vec \sigma). \tag{4.12}$$

If we apply $\sum_a\, \partial_a$ to Eqs.(4.12) we get

$${4\over 3}\, \triangle\, \Big[\sum_a\, \partial_a\, \bar n_{(1)(a)}\Big](\tau ,\vec \sigma) \,{\buildrel \circ \over =}\, \Big[{4\over 3}\, \triangle\, {}^3K_{(1)} +$$
$$ + {{16\pi\, G}\over {c^3}}\, \sum_a\, \partial_a\, {\cal M}^{(UV)}_{(1)a} + 2\, \sum_a\, \partial_{\tau}\, \partial_a^2\, \Gamma_a^{(1)}\Big](\tau ,\vec \sigma), \tag{4.13}$$

and this equation implies

$$\sum_a\, \partial_a\, \bar n_{(1)(a)}(\tau ,\vec \sigma) \,{\buildrel \circ \over =}\, \Big[{}^3K_{(1)} + {{12\pi\, G}\over {c^3}}\, {1\over {\triangle}}\, \sum_a\, \partial_a\, {\cal M}^{(UV)}_{(1)a} + {3\over 2}\, {1\over {\triangle}}\, \sum_a\, \partial_{\tau}\, \partial_a^2\, \Gamma_a^{(1)}\Big](\tau ,\vec \sigma). \tag{4.14}$$

As a consequence the final linearized equation (of elliptic type) for the shift functions is

$$\begin{aligned} \triangle\, \bar n_{(1)(a)}(\tau ,\vec \sigma) \,\, &{\buildrel \circ \over =}\,\, \Big[\partial_a\, {}^3K_{(1)} + {{4\pi\, G}\over {c^3}}\, \Big(4\, {\cal M}^{(UV)}_{(1)a} - {{\partial_a}\over {\triangle}}\, \sum_c\, \partial_c\, {\cal M}^{(UV)}_{(1)c}\Big) +\\ &+\,\, {1\over 2}\, \Big(4\, \partial_{\tau}\, (\partial_a\, \Gamma_a^{(1)}) - {{\partial_a}\over {\triangle}}\, \sum_c\, \partial_{\tau}\, (\partial_c^2\, \Gamma_c^{(1)})\Big)\Big](\tau ,\vec \sigma), \end{aligned} \tag{4.15}$$

whose solution, vanishing at spatial infinity and depending upon the tidal variables, is

$$
\begin{aligned}
\bar{n}_{(1)(a)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[\frac{\partial_a}{\triangle}\, {}^3K_{(1)} + \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\Big(4\, \mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a}{\triangle}\, \sum_c \partial_c\, \mathcal{M}^{(UV)}_{(1)c}\Big) + \\
&+\ \frac{1}{2}\, \partial_\tau\, \frac{\partial_a}{\triangle}\Big(4\, \Gamma_a^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma_c^{(1)}\Big)\Big](\tau,\vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=}\ -\frac{4\, G}{c^3}\, \sum_i \eta_i\, \Big[\frac{\kappa_{ia}(\tau) - \frac{Q_i}{c}\, A_{\perp a}(\tau,\vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} - \\
&-\ \sum_c \Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\, A_{\perp c}(\tau,\vec{\eta}_i(\tau))\Big) \int \frac{d^3\sigma_1}{4\pi\, |\vec{\sigma} - \vec{\sigma}_1|\, |\vec{\sigma}_1 - \vec{\eta}_i(\tau)|^3} \\
&\quad \Big(\delta_{ac} - 3\, \frac{(\sigma_1^a - \eta_i^a(\tau))\, (\sigma_1^c - \eta_i^c(\tau))}{|\vec{\sigma}_1 - \vec{\eta}_i(\tau)|^2}\Big)\Big] - \\
&-\ \int \frac{d^3\sigma_1}{4\pi\, |\vec{\sigma} - \vec{\sigma}_1|}\, \partial_{1a}\, \Big[\Big({}^3K_{(1)} + 2\, \partial_\tau\, \Gamma_a^{(1)}\Big)(\tau,\vec{\sigma}_1) + \\
&+\ \int d^3\sigma_2\, \frac{\sum_c \partial_\tau\, \partial_{2c}^2\, \Gamma_c^{(1)}(\tau,\vec{\sigma}_2)}{8\pi\, |\vec{\sigma}_1 - \vec{\sigma}_2|}\Big] + \\
&+\ \frac{G}{c^4} \int \frac{d^3\sigma_1}{|\vec{\sigma} - \vec{\sigma}_1|}\, \sum_s \\
&\quad \Big[4\, F_{as}(\tau,\vec{\sigma}_1)\, \Big(\pi_\perp^s(\tau,\vec{\sigma}_1) - \sum_n \delta^{sn}\, \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}_1,\vec{\eta}_i(\tau))}{\partial\, \sigma_1^n}\Big) + \\
&+\ \sum_c \partial_{1a}\, \partial_{1c} \int \frac{d^3\sigma_2}{4\pi\, |\vec{\sigma}_1 - \vec{\sigma}_2|} \\
&\quad F_{cs}(\tau,\vec{\sigma}_2)\, \Big(\pi_\perp^s(\tau,\vec{\sigma}_2) - \sum_n \delta^{sn}\, \sum_i Q_i\, \eta_i\, \frac{\partial\, c(\vec{\sigma}_2,\vec{\eta}_i(\tau))}{\partial\, \sigma_2^n}\Big)\Big], \\
{}^4g_{(1)\tau r}(\tau,\vec{\sigma}) &\stackrel{\circ}{=}\ -\epsilon\, \bar{n}_{(1)(r)}(\tau,\vec{\sigma}).
\end{aligned}
\qquad (4.16)
$$

Then the functions $\sigma_{(1)(a)(b)}|_{a\not= b}$ have the the following expression

$$
\begin{aligned}
\sigma_{(1)(a)(b)}|_{a\not= b} &\stackrel{\circ}{=}\ \frac{1}{2}\, \Big(\partial_a\, \bar{n}_{(1)(b)} + \partial_b\, \bar{n}_{(1)(a)}\Big)(\tau,\vec{\sigma}) \stackrel{\circ}{=} \\
&\stackrel{\circ}{=}\ \frac{\partial_a\, \partial_b}{\triangle}\, {}^3K_{(1)} + \frac{8\pi\, G}{c^3}\, \Big[\frac{1}{\triangle}\, \Big(\partial_a\, \mathcal{M}^{(UV)}_{(1)b} + \partial_b\, \mathcal{M}^{(UV)}_{(1)a}\Big) - \frac{1}{2}\, \frac{\partial_a\, \partial_b}{\triangle}\, \sum_c \frac{\partial_c}{\triangle}\, \mathcal{M}^{(UV)}_{(1)c}\Big] + \\
&+\ \partial_\tau\, \frac{\partial_a\, \partial_b}{\triangle}\, \Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2}\, \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma_c^{(1)}\Big) \stackrel{\circ}{=}
\end{aligned}
$$

$$
\begin{aligned}
&\circeq -\frac{1}{2}\sum_d (\delta_{ad}\,\partial_b + \delta_{bd}\,\partial_a)\quad \Big(\frac{4\,G}{c^3}\sum_i \eta_i\,\Big[\frac{\kappa_{id}(\tau) - \frac{Q_i}{c}\,A_{\perp d}(\tau, \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} -\\
&-\sum_c \Big(\kappa_{ic}(\tau) - \frac{Q_i}{c}\,A_{\perp c}(\tau, \vec{\eta}_i(\tau))\Big) \int \frac{d^3\sigma_1}{4\pi\,|\vec{\sigma} - \vec{\sigma}_1|\,|\vec{\sigma}_1 - \vec{\eta}_i(\tau)|^3}\\
&\Big(\delta_{dc} - 3\,\frac{(\sigma_1^d - \eta_i^d(\tau))\,(\sigma_1^c - \eta_i^c(\tau))}{|\vec{\sigma}_1 - \vec{\eta}_i(\tau)|^2}\Big)\Big] +\\
&+\int \frac{d^3\sigma_1}{4\pi\,|\vec{\sigma} - \vec{\sigma}_1|}\,\partial_{1d}\,\Big[\Big({}^3K_{(1)} + 2\,\partial_\tau\,\Gamma_d^{(1)}\Big)(\tau, \vec{\sigma}_1) + \int d^3\sigma_2\,\frac{\sum_c \partial_\tau\,\partial_{2c}^2\,\Gamma_c^{(1)}(\tau, \vec{\sigma}_2)}{8\pi\,|\vec{\sigma}_1 - \vec{\sigma}_2|}\Big]\ \Big) +\\
&\\
&+\frac{G}{2\,c^4}\Big(\partial_a \int \frac{d^3\sigma_1}{|\vec{\sigma} - \vec{\sigma}_1|}\sum_s \Big[4\,F_{bs}(\tau, \vec{\sigma}_1)\Big(\pi_\perp^s(\tau, \vec{\sigma}_1) - \sum_n \delta^{sn}\sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma}_1, \vec{\eta}_i(\tau))}{\partial\,\sigma_1^n}\Big) +\\
&+\sum_c \partial_{1b}\,\partial_{1c}\int \frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_1 - \vec{\sigma}_2|}\,F_{cs}(\tau, \vec{\sigma}_2)\Big(\pi_\perp^s(\tau, \vec{\sigma}_2) - \sum_n \delta^{sn}\sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma}_2, \vec{\eta}_i(\tau))}{\partial\,\sigma_2^n}\Big)\Big] +\\
&+\partial_b \int \frac{d^3\sigma_1}{|\vec{\sigma} - \vec{\sigma}_1|}\sum_s \Big[4\,F_{as}(\tau, \vec{\sigma}_1)\Big(\pi_\perp^s(\tau, \vec{\sigma}_1) - \sum_n \delta^{sn}\sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma}_1, \vec{\eta}_i(\tau))}{\partial\,\sigma_1^n}\Big) +\\
&+\sum_c \partial_{1a}\,\partial_{1c}\int \frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_1 - \vec{\sigma}_2|}\,F_{cs}(\tau, \vec{\sigma}_2)\Big(\pi_\perp^s(\tau, \vec{\sigma}_2) - \sum_n \delta^{sn}\sum_i Q_i\,\eta_i\,\frac{\partial\,c(\vec{\sigma}_2, \vec{\eta}_i(\tau))}{\partial\,\sigma_2^n}\Big)\Big]\Big).
\end{aligned}
\tag{4.17}
$$

Both the functions $\bar{n}_{(1)(a)}$ and $\sigma_{(1)(a)(b)}|_{a\neq b}$ depend upon matter, upon the tidal variables and upon the spatial gradients of the inertial gauge function ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$.

Due to Eq.(4.16) we have that gravito-magnetism (described by ${}^4g_{(1)\tau r}$) in the 3-orthogonal gauges depends on both the instantaneous inertial and retarded parts of the gravitational field and upon the gauge variable $\partial_r\,{}^3\mathcal{K}_{(1)}$.

The integral appearing in the shift function has the following expression $\int \frac{d^3\sigma_1}{4\pi\,|\vec{\sigma}-\vec{\sigma}_1|\,|\vec{\sigma}_1-\vec{\eta}_i(\tau)|^3}\Big(\delta^{ac} - 3\,\frac{(\sigma_1^a-\eta_i^a(\tau))\,(\sigma_1^c-\eta_i^c(\tau))}{|\vec{\sigma}_1-\vec{\eta}_i(\tau)|^2}\Big) = -\frac{1}{2}\,\frac{1}{|\vec{\sigma}-\vec{\eta}_i(\tau)|}\Big(\delta^{ac} - \frac{(\sigma^a-\eta_i^a(\tau))\,(\sigma^c-\eta_i^c(\tau))}{|\vec{\sigma}-\vec{\eta}_i(\tau)|^2}\Big)$, so that the contribution to gravito-magnetism coming from the mass current density $\mathcal{M}^{(UV)}_{(1)(r)}(\tau, \vec{\sigma})$ has the final form $-\frac{2\,G}{c^3}\sum_i \frac{\eta_i}{|\vec{\sigma}-\vec{\eta}_i(\tau)|}\Big(\kappa_{ia}(\tau) - \frac{Q_i}{c}\,A_{\perp a}(\tau, \vec{\eta}_i(\tau)) + \frac{(\sigma^a-\eta_i^a(\tau))\,\Big[\vec{\kappa}_i(\tau)-\frac{Q_i}{c}\,\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))\Big]\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{|\vec{\sigma}-\vec{\eta}_i(\tau)|^2}\Big)$.

From Eqs.(3.6) and (4.16) we get that the tidal momenta $\Pi_{\bar{a}}$ have the following expression

$$
\begin{aligned}
\frac{8\pi G}{c^3}\,\Pi_{\bar a}(\tau,\vec\sigma) &\stackrel{\circ}{=} \partial_\tau\, R_{\bar a}(\tau,\vec\sigma) - \sum_a \gamma_{\bar a a}\,\Big[\partial_\tau\,\frac{\partial_a^2}{2\,\triangle}\,(4\,\Gamma_a^{(1)} - \frac{1}{\triangle}\sum_c \partial_c^2\,\Gamma_c^{(1)}) +\\
&+\ \frac{4\pi G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a^2}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}) + \frac{\partial_a^2}{\triangle}\,{}^3K_{(1)}\Big] =\\
&= \Big(\sum_{\bar b} M_{\bar a\bar b}\,\partial_\tau\, R_{\bar b} - \sum_a \gamma_{\bar a a}\,\Big[\frac{4\pi G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a^2}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}) +\\
&+\ \frac{\partial_a^2}{\triangle}\,{}^3K_{(1)}\Big]\Big)(\tau,\vec\sigma),\\
&\quad M_{\bar a\bar b} = \delta_{\bar a\bar b} - \sum_a \gamma_{\bar a a}\,\frac{\partial_a^2}{\triangle}\,\Big(2\,\gamma_{\bar b a} - \frac{1}{2}\sum_b \gamma_{\bar b b}\,\frac{\partial_b^2}{\triangle}\Big),
\end{aligned}
\qquad (4.18)
$$

where we introduced the operator $M_{\bar a\bar b}$, which will be shown to be connected with the selection of the traceless-transverse (TT) part of the 3-metric on Σ_τ in Section VI. The relation between tidal momenta and tidal velocities depends on the inertial gauge variable $\sum_a \gamma_{\bar a a}\,\partial_a^2\,{}^3\mathcal{K}$.

C. The Contracted Bianchi Identities

The contracted Bianchi identity (2.8) for $\partial_\tau\,\phi_{(1)}$ has the following linearized form

$$
\partial_\tau\,\phi_{(1)}(\tau,\vec\sigma)\ = 6\,\partial_\tau\,q_{(1)}(\tau,\vec\sigma) \stackrel{\circ}{=} \frac{1}{6}\,\Big(\sum_a \partial_a\,\bar n_{(1)(a)} - {}^3K_{(1)}\Big)(\tau,\vec\sigma) + O(\zeta^2). \qquad (4.19)
$$

By using the solutions (4.6) and (4.16) for $\phi_{(1)}$ and $\bar n_{(1)(a)}$, this equation is identically satisfied at the lower order due to the conservation $\partial_A\,T^{A\tau}_{(1)} = 0$ at the lowest order, see Eq.(3.13).

The Bianchi identities (2.9) become

$$
\partial_\tau\,\pi_i^{(\theta)} \stackrel{\circ}{=} 0 + O(\zeta^2), \qquad (4.20)
$$

consistently with Eq. (3.7) , which gives $\pi_i^{(\theta)}(\tau,\vec\sigma) = 0 + O(\zeta^2)$ in our 3-orthogonal gauges.

It can also be checked with a lengthy calculation that also the contracted Bianchi identities (2.10) for $\sigma_{(1)(a)(b)}|_{a\neq b}$ are satisfied at this order. This check requires the use of Eqs. (4.8), (3.6), (4.9), (4.17), (4.6), (4.7), (3.13) (i.e. of the already found solutions of the linearized Hamilton equations) to get an expression which vanishes if we use Eqs.(6.11), which are a byproduct of the linearized second order equations (6.4) for the tidal variables as shown in Section VI.

As a consequence possible problems with the generalized gravitational Gribov problem, identified in Ref.[6], will appear at higher orders.

Therefore the contracted Bianchi identities are identically satisfied at the lowest order.

D. The ADM Poincare' Generators

The weak ADM Poincare' generators in the family of 3-orthogonal gauges are given in Eqs. (2.14) and (2.20), with the mass and momentum densities of Eqs.(2.4). They are the analogue of the internal Poincare' generators of Minkowski inertial rest-frame instant form [4].

By using Eq.(3.6) for the tidal momenta and Eqs. (2.4), (2.5) and (2.14), after a long but straightforward calculation (including various integrations by parts) we get the following form of the weak ADM energy at the second order [8]

$$
\begin{aligned}
\frac{1}{c}\,\hat{E}_{ADM} &= \int d^3\sigma\,\Big[\mathcal{M}^{(UV)}_{(1)} + \mathcal{M}^{(UV)}_{(2)} + \\
&+ \frac{c^3}{16\pi\,G}\Big(\sum_{\bar a}(\partial_\tau\,R_{\bar a} - \sum_a \gamma_{\bar a a}\,\partial_a\,\bar n_{(1)(a)})^2 + \sum_{a\neq b}\sigma^2_{(1)(a)(b)} - \\
&- \frac{2}{3}\,({}^3K_{(1)})^2 - \mathcal{S}_{(2)}|_{\theta^i=0}\Big](\tau,\vec\sigma) + O(\zeta^3) = \\
&= M_{(1)}\,c + \frac{1}{c}\,\hat{E}_{ADM(2)} + M\,c\,O(\zeta^3),\\
&\quad \mathcal{S}_{(2)}|_{\theta^i=0} = \sum_a\,\Big[8\,(\partial_a\,\phi_{(1)})^2 - 4\sum_{\bar a}\gamma_{\bar a a}\,\partial_a\,\phi_{(1)}\,\partial_a\,R_{\bar a} + \\
&\quad + \sum_{\bar a\bar b}(2\,\gamma_{\bar a a}\,\gamma_{\bar b a} - \delta_{\bar a\bar b})\,\partial_a\,R_{\bar a}\,\partial_a\,R_{\bar b}\Big] + O(\zeta^3),
\end{aligned}
$$

$$
\begin{aligned}
M_{(1)}\,c &= \int d^3\sigma\,\mathcal{M}^{(UV)}_{(1)}(\tau,\vec\sigma) = \sum_i \eta_i\,\sqrt{m_i^2\,c^2 + \Big(\vec\kappa_i(\tau) - \frac{Q_i}{c}\,\vec A_\perp(\tau,\vec\eta_i(\tau))\Big)^2} + \\
&+ \frac{1}{2c}\int d^3\sigma\,\Big(\Big[\sum_a\,\Big((\pi^a_\perp)^2 - \Big(2\,\pi^a - \sum_i Q_i\,\eta_i\,\frac{c(\vec\sigma,\vec\eta_i(\tau))}{\partial\,\sigma^a}\Big)\\
&\sum_j Q_j\,\eta_j\,\frac{c(\vec\sigma,\vec\eta_j(\tau))}{\partial\,\sigma^a}\Big) + \frac{1}{2}\sum_{ab} F^2_{ab}\Big]\Big)(\tau,\vec\sigma),
\end{aligned}
$$

[8] While the terms $\sum_a \gamma_{\bar a a}\,\partial_a\,\bar n_{(1)(a)}$ coming from the tidal momenta $\Pi_{\bar a}$ are *gravito-magnetic* potentials of the 3-orthogonal gauges (they depend on the shift functions), the term $\sum_{a\neq b}\sigma^2_{(1)(a)(b)}$, determined by the super-momentum constraints, is a 3-coordinate dependent potential like $\mathcal{S}_{(2)}$ (coming from the Gamma-Gamma term of the 3-curvature).

$$
\begin{aligned}
\frac{1}{c}\,\hat{E}_{ADM(2)} \;&=\; \int d^3\sigma\,\Big(\mathcal{M}^{(UV)}_{(2)} + \Big(\sum_a \partial_a\,\mathcal{M}^{(UV)}_{(1)a}\Big)\,\frac{1}{\triangle}\,{}^3K_{(1)} +\\
&+\; \frac{8\pi\,G}{c^3}\,\Big[\frac{1}{4}\,\mathcal{M}^{(UV)}_{(1)}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} - \sum_a \mathcal{M}^{(UV)}_{(1)a}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)a} - \frac{1}{4}\,\Big(\sum_a \frac{\partial_a}{\triangle}\,\mathcal{M}^{(UV)}_{(1)a}\Big)^2\Big] +\\
&+\; \frac{c^3}{16\pi G}\,\sum_{\bar a\bar b}\,\Big[\partial_\tau\,R_{\bar a}\,M_{\bar a\bar b}\,\partial_\tau\,R_{\bar b} + \sum_a\,\partial_a\,R_{\bar a}\,M_{\bar a\bar b}\,\partial_a\,R_{\bar b}\Big]\Big)(\tau,\vec\sigma).
\end{aligned}
\tag{4.21}
$$

Eqs. (4.6), (4.16) and (4.17) for $\phi_{(1)}$, $\bar n_{(1)(a)}$ and $\sigma_{(1)(a)(b)}|_{a\neq b}$ have been used to get $\hat{E}_{ADM(2)}$. Since, as we will see in Section VII, the solution of the Hamilton equation for the tidal variables $R_{\bar a}$ is proportional to $\frac{G}{c^3}$, we see that all the terms not in the first line of $\hat{E}_{(2)ADM}$ are of order $\frac{G}{c^3}$.

Eq.(3.13) implies $\partial_\tau\,M_{(1)}c = 0 + \frac{Mc}{L}\,O(\zeta^2)$.

In $\hat{E}_{ADM(2)}$ we can see a kinetic term for the tidal variables $R_{\bar a}$, with the operator $M_{\bar a\bar b}$ appearing in Eq.(4.18) (the connected operator $\tilde{M}_{ab}$ appearing in the boosts (4.37) is defined in Eq.(6.4) of Section VI). Moreover, there is a potential for the tidal variables and bilinear terms in the matter. Finally, there is a term depending on the inertial gauge variable (non-local York time) ${}^3\mathcal{K} = \frac{1}{\triangle}\,{}^3K_{(1)}$ coupled to the divergence of the matter current density. However at this order the *negative definite quadratic term in the local York time* ${}^3K_{(1)} = \triangle\,{}^3\mathcal{K}_{(1)}$ *appearing* in Eq.(2.14) disappears due to the elimination of the tidal momenta with Eq.(4.18), the use of the solution of the constraints and of some of the Hamilton equations plus integrations by parts. As we will see in Eqs. (4.23) and (4.37) in the six Lorentz generators there is a dependence both on the local and non-local York time at the second order.

For the other weak Poincare' generators the weak field approximation of Eqs.(2.20), by using Eqs.(4.18) and by making integrations by parts, gives (all the terms in $p^r_{(2)}$, $j^{rs}_{(2)}$ and all the terms except the first in $j^{\tau r}_{(2)}$ are of order $\frac{G}{c^3}$ on the solution for the tidal variables given in Section VII; the last term in the boost generators is a surface term which can be dropped with our boundary conditions)

$$
\hat{P}^r_{ADM} \;=\; p^r_{(1)} + p^r_{(2)} + Mc\,O(\zeta^3) \approx 0,
$$

$$
\begin{aligned}
p^r_{(1)} \;&=\; \int d^3\sigma\,\mathcal{M}^{(UV)}_{(1)r}(\tau,\vec\sigma) = \sum_i\,\eta_i\,\Big(\kappa_{ir}(\tau) - \frac{Q_i}{c}\,A_{\perp r}(\tau,\vec\eta_i(\tau))\Big) -\\
&-\; \frac{1}{c}\,\int d^3\sigma\,\sum_s\,F_{rs}(\tau,\vec\sigma)\,\Big(\pi^s_\perp(\tau,\vec\sigma) - \sum_n\,\delta^{sn}\,\sum_i\,Q_i\,\eta_i\,\frac{\partial\,c(\vec\sigma,\vec\eta_i(\tau))}{\partial\,\sigma^n}\Big),
\end{aligned}
$$

$$
\begin{aligned}
p^r_{(2)} \;&=\; -\int d^3\sigma\, \Big(\frac{c^3}{8\pi\, G}\, \sum_{\bar a\bar b}\, \partial_r\, R_{\bar a}\, M_{\bar a\bar b}\, \partial_\tau\, R_{\bar b} + \\
&+\; \sum_a\, {\cal M}^{(UV)}_{(1)a}\, \frac{\partial_r\, \partial_a}{\triangle}\, \Big(2\, \Gamma^{(1)}_a - \frac{1}{2}\, \sum_c\, \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c \Big) - \\
&-\; {\cal M}^{(UV)}_{(1)}\, \frac{\partial_r}{\triangle}\, {}^3K_{(1)} \Big)(\tau ,\vec \sigma),
\end{aligned} \tag{4.22}
$$

$$
\hat J^{rs}_{ADM} \;=\; j^{rs}_{(1)} + j^{rs}_{(2)} + M c\, O(\zeta^3),
$$

$$
\begin{aligned}
j^{rs}_{(1)} &= \int d^3\sigma\, \Big(\sigma^r\, {\cal M}^{(UV)}_{(1)s} - \sigma^s\, {\cal M}^{(UV)}_{(1)r} \Big)(\tau ,\vec \sigma) = \\
&=\; \sum_i\, \eta_i\, \Big[\eta^r_i(\tau)\, \Big(\kappa_{is}(\tau) - \frac{Q_i}{c}\, A_{\perp s}(\tau ,\vec \sigma) \Big) - \eta^s_i(\tau)\, \Big(\kappa_{ir}(\tau) - \frac{Q_i}{c}\, A_{\perp r}(\tau ,\vec \sigma) \Big) \Big] - \\
&-\; \frac{1}{c}\, \int d^3\sigma\, \Big[\sigma^r\, \sum_u\, F_{su}(\tau ,\vec \sigma)\, \Big(\pi^u_{\perp}(\tau ,\vec \sigma) - \sum_n\, \delta^{un}\, \sum_i\, Q_i\, \eta_i\, \frac{\partial\, c(\vec \sigma , \vec \eta_i(\tau))}{\partial\, \sigma^n} \Big) - \\
&-\; \sigma^s\, \sum_u\, F_{ru}(\tau ,\vec \sigma)\, \Big(\pi^u_{\perp}(\tau ,\vec \sigma) - \sum_n\, \delta^{un}\, \sum_i\, Q_i\, \eta_i\, \frac{\partial\, c(\vec \sigma , \vec \eta_i(\tau))}{\partial\, \sigma^n} \Big) \Big],
\end{aligned}
$$

$$
\begin{aligned}
j^{rs}_{(2)} \;&=\; \int d^3\sigma\, \Big({\cal M}^{(UV)}_{(1)}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{1}{\triangle}\, {}^3K_{(1)} - \\
&-\; 2\, \sum_u\, {\cal M}^{(UV)}_{(1)u}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{\partial_u}{\triangle}\, \Gamma^{(1)}_u - \frac{1}{2}\, \sum_c\, ({\cal M}^{(UV)}_{(1)r}\, \frac{\partial_s}{\triangle} - {\cal M}^{(UV)}_{(1)s}\, \frac{\partial_r}{\triangle})\, \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c + \\
&+\; \frac{1}{2}\, \sum_{uv}\, {\cal M}^{(UV)}_{(1)u}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{\partial_u}{\triangle}\, \frac{\partial_v^2}{\triangle}\, \Gamma^{(1)}_v + 2\, \Big({\cal M}^{(UV)}_{(1)r}\, \frac{\partial_s}{\triangle}\, \Gamma^{(1)}_s - {\cal M}^{(UV)}_{(1)s}\, \frac{\partial_r}{\triangle}\, \Gamma^{(1)}_r \Big) - \\
&-\; \frac{c^3}{8\pi\, G}\, \Big[\sum_{\bar a\bar b}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, R_{\bar a}\, M_{\bar a\bar b}\, \partial_\tau\, R_{\bar b} + 2\, {}^3K_{(1)}\, \partial_r\, \partial_s\, (\Gamma^{(1)}_s - \Gamma^{(1)}_r) + \\
&+\; 2\, \partial_\tau\, (\Gamma^{(1)}_r + \Gamma^{(1)}_s - \frac{1}{2}\, \sum_c\, \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c)\, \frac{\partial_r\, \partial_s}{\triangle}\, (\Gamma^{(1)}_s - \Gamma^{(1)}_r) \Big] \Big)(\tau ,\vec \sigma),
\end{aligned} \tag{4.23}
$$

$$
\hat J^{\tau r}_{ADM} \;\;=\; - \hat J^{r\tau}_{ADM} = j^{\tau r}_{(1)} + j^{\tau r}_{(2)} + M c\, O(\zeta^3) \approx 0,
$$

$$
j^{\tau r}_{(1)} = - j^{r\tau}_{(1)} = - \int d^3\sigma\, \sigma^r\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) = - \sum_i \eta_i\, \eta_i^r\, \sqrt{m_i^2\, c^2 + \Big(\vec{\kappa}_i(\tau) - \frac{Q_i}{c}\, \vec{A}_{\perp}(\tau, \vec{\eta}_i(\tau))\Big)^2} +
$$

$$
+\ \frac{1}{2c} \int d^3\sigma\, \sigma^r\, \Big(\Big[\sum_a \Big((\pi^a_{\perp})^2 - \Big(2\, \pi^a - \sum_i Q_i\, \eta_i\, \frac{c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial\, \sigma^a}\Big)
$$

$$
\sum_j Q_j\, \eta_j\, \frac{c(\vec{\sigma}, \vec{\eta}_j(\tau))}{\partial\, \sigma^a}\Big) + \frac{1}{2} \sum_{ab} F^2_{ab}\Big]\Big)(\tau, \vec{\sigma}),
$$

$$
\begin{aligned}
j^{\tau r}_{(2)} &= -\int d^3\sigma\, \sigma^r\, \mathcal{M}^{(UV)}_{(2)}(\tau,\vec{\sigma}) + && (4.24)\\
&- \int d^3\sigma\, \sigma^r \Big\{ \frac{c^3}{16\pi G} \sum_{a,b} (\partial_a \Gamma_b^{(1)})^2 - \frac{c^3}{8\pi G} \sum_a \partial_a \Gamma_a^{(1)}\, \partial_a \Big(\Gamma_a^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) - && (4.25)\\
&- \frac{c^3}{32\pi G} \sum_a \partial_a \Big(\sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) \partial_a \Big(\sum_d \frac{\partial_d^2 \Gamma_d^{(1)}}{\Delta} \Big) + && (4.26)\\
&+ \frac{1}{2} \sum_a \frac{\partial_a \mathcal{M}^{UV}_{(1)}}{\Delta}\, \partial_a \Big(\Gamma_a^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) + \frac{32\pi G}{c^3} \sum_a \Big(\frac{\partial_a \mathcal{M}^{UV}_{(1)}}{\Delta} \Big)^2 + && (4.27)\\
&+ \frac{c^3}{16\pi G} \sum_{a,b} \Big(\widetilde{M}_{ab}\, \partial_\tau \Gamma_b^{(1)} \Big)^2 + \frac{c^3}{16\pi G} \sum_{a\neq b} \Big[\frac{\partial_a\partial_b\partial_\tau}{\Delta} \Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) \Big]^2 - && (4.28)\\
&- 2 \sum_{a,b} \Big(\widetilde{M}_{ab}\, \partial_\tau \Gamma_b^{(1)} \Big) \frac{\partial_a \mathcal{M}^{UV}_{(1)a}}{\Delta} + 2 \sum_{a\neq b} \frac{\partial_a\partial_b\partial_\tau}{\Delta} \Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) \frac{\partial_a \mathcal{M}^{UV}_{(1)a}}{\Delta} - && (4.29)\\
&- \frac{c^3}{8\pi G} \sum_{a,b} \Big(\widetilde{M}_{ab}\, \partial_\tau \Gamma_b^{(1)} \Big) \frac{\partial_a^2}{\Delta} \Big({}^3K_{(1)} - \frac{4\pi G}{c^3} \sum_c \frac{\partial_c \mathcal{M}^{UV}_{(1)c}}{\Delta} \Big) + && (4.30)\\
&+ \frac{c^3}{8\pi G} \sum_{a\neq b} \frac{\partial_a\partial_b\partial_\tau}{\Delta} \Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) \frac{\partial_a\partial_b}{\Delta} \Big({}^3K_{(1)} - \frac{4\pi G}{c^3} \sum_c \frac{\partial_c \mathcal{M}^{UV}_{(1)c}}{\Delta} \Big) + && (4.31)\\
&+ \frac{c^3}{16\pi G} \sum_{a,b} \Big[\frac{\partial_a\partial_b}{\Delta} \Big({}^3K_{(1)} - \frac{4\pi G}{c^3} \sum_c \frac{\partial_c \mathcal{M}^{UV}_{(1)c}}{\Delta} \Big) \Big]^2 + && (4.32)\\
&+ 2 \sum_{a,b} \frac{\partial_a \mathcal{M}^{UV}_{(1)b}}{\Delta} \frac{\partial_a\partial_b}{\Delta} \Big({}^3K_{(1)} - \frac{4\pi G}{c^3} \sum_c \frac{\partial_c \mathcal{M}^{UV}_{(1)c}}{\Delta} \Big) - && (4.33)\\
&- \frac{c^3}{48\pi G} \Big({}^3K_{(1)} + \frac{12\pi G}{c^3} \sum_c \frac{\partial_c \mathcal{M}^{UV}_{(1)c}}{\Delta} \Big)^2 - \frac{c^3}{24\pi G} \Big({}^3K_{(1)} \Big)^2 + && (4.34)\\
&+ \frac{8\pi G}{c^3} \sum_{a,b} \Big[\Big(\frac{\partial_a \mathcal{M}^{UV}_{(1)b}}{\Delta} \Big)^2 + \frac{\partial_a \mathcal{M}^{UV}_{(1)b}}{\Delta} \frac{\partial_b\, \mathcal{M}^{UV}_{(1)a}}{\Delta} \Big] \Big\}(\tau,\vec{\sigma}) + && (4.35)\\
&+ \int d^3\sigma\, \Big[-\frac{3}{2} \frac{\mathcal{M}^{UV}_{(1)}}{\Delta}\, \partial_r \Gamma_r^{(1)} + \frac{3c^3}{16\pi G} \partial_r \Gamma_r^{(1)} \Big(\sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big) \Big](\tau,\vec{\sigma}) + && (4.36)\\
&- \int d^3\sigma\, \partial_r \Big\{ \frac{c^3}{16\pi G} \Big[2\Big(\Gamma_r^{(1)} \Big)^2 - \sum_s \Big(\Gamma_s^{(1)} \Big)^2 - \frac{1}{2} \Big(\sum_c \frac{\partial_c^2 \Gamma_c^{(1)}}{\Delta} \Big)^2 \Big] - \frac{2\pi G}{c^3} \Big(\frac{\mathcal{M}_{(1)}}{\Delta} \Big)^2 \Big\}(\tau,\vec{\sigma}). && (4.37)
\end{aligned}
$$

We see that at the lowest order we get an unfaithful realization of the ten *internal* Poincare' generators $M_{(1)}\, c$, $p^r_{(1)} \approx 0 + O(\zeta^2)$, $j^{rs}_{(1)}$, $j^{\tau r}_{(1)} \approx 0 + O(\zeta^2)$ for the same matter in a rest-frame instant form approximately valid [modulo corrections at $O(\zeta^2)$] in an abstract Minkowski space-time with the Wigner instantaneous 3-spaces coinciding with the

asymptotic inertial 3-spaces at spatial infinity of our space-times. They are the *internal* Poincare' generators of the system positive -energy charged scalar particles plus transverse electro-magnetic field in the radiation gauge studied in Refs.[12, 13].

Like in Minkowski space-time the internal 3-center of mass $\vec{\eta}(\tau)$ of the isolated system *3-universe* (i.e. "gravitational field plus particles plus electro-magnetic field" on the instantaneous 3-space Σ_τ) and its conjugate momentum are eliminated by the constraints $p^r_{(1)} \approx 0 + O(\zeta^2)$ and $j^{\tau r}_{(1)} \stackrel{def}{=} - M_{(1)}\, c\, \eta^r(\tau) \approx 0 + O(\zeta^2)$. They imply $\eta^r(\tau) = -\frac{1}{M_{(1)}\, c} \int d^3\sigma\, \sigma^r\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) \approx 0$, i.e. the 3-center of mass of the 3-universe is in the origin of the 3-coordinates: the world-line of the time-like observer is the Fokker-Pryce 4-center of inertia $Y^\mu(\tau) = z^\mu(\tau, \vec{0})$ [4, 12, 13], to be interpreted as a version in the bulk of an asymptotic inertial observer.

With the methods used in Refs.[4, 12, 13] it could be shown that in our asymptotically Minkowskian space-times there is a decoupled (Newton-Wigner) canonical 4-center of mass $\tilde{x}^\mu_{com}(\tau) = z^\mu(\tau, \tilde{\vec{\eta}}(\tau))$ of the instantaneous 3-universes Σ_τ, carrying a pole-dipole structure (mass $\frac{1}{c}\, \hat{E}_{ADM}$ and rest spin $\hat{J}^{rs}_{ADM}$) and non-covariant with respect to an *external* asymptotic ADM Poincare' group with the same generators as in special relativity [4, 12, 13].

E. The Effective Hamiltonian in the 3-Orthogonal Gauge with a Given ${}^3K_{(1)}(\tau, \vec{\sigma})$

The restriction of the Hamilton equations of paper I to our family of 3-orthogonal gauges, given in Section II, can also be generated by using an effective Hamiltonian determined by the gauge-fixing procedure.

Since our gauge fixings are $\theta^i(\tau, \vec{\sigma}) \approx 0$ and $\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) - \frac{c^3}{12\pi\, G}\, F(\tau, \vec{\sigma}) \approx 0$ with ${}^3K(\tau, \vec{\sigma}) \approx F(\tau, \vec{\sigma})$ an arbitrary numerical function, we must take into account the explicit τ-dependence of this function which implies that the effective Hamiltonian is no more $\frac{1}{c}\, \hat{E}_{ADM}$ with the weak ADM energy given in Eq.(4.21).

To find the effective Hamiltonian we should perform the following time-dependent canonical transformation: $\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) \mapsto \pi^{'}_{\tilde{\phi}}(\tau, \vec{\sigma}) = \pi_{\tilde{\phi}}(\tau, \vec{\sigma}) - \frac{c^3}{12\pi\, G}\, F(\tau, \vec{\sigma})$, $\tilde{\phi}^{'}(\tau, \vec{\sigma}) = \tilde{\phi}(\tau, \vec{\sigma})$, with all the other canonical variables fixed. Due to the explicit time-dependence the new effective Dirac Hamiltonian is $H^{'}_D = H_D + \frac{c^3}{12\pi\, G} \int d^3\sigma\, \frac{\partial\, F(\tau, \vec{\sigma})}{\partial\, \tau} \tilde{\phi}(\tau, \vec{\sigma})$. In the new canonical basis the gauge fixings are $\theta^i(\tau, \vec{\sigma}) \approx 0$, $\pi^{'}_{\tilde{\phi}}(\tau, \vec{\sigma}) \approx 0$ and have no explicit τ-dependence.

However to go to the reduced phase space we have to find the Dirac brackets and this would require the explicit solution of the super-Hamiltonian and super-momentum constraints. This can be done in the linearized theory.

From Eq.(4.18) we get (see Section VI for the inverse of the operator $M_{\bar{a}\bar{b}}$)

$$\partial_\tau\, R_{\bar{a}} = \sum_{\bar{b}}\, M^{-1}_{\bar{a}\bar{b}}\, \Big(\frac{8\pi\, G}{c^3}\, \Pi_{\bar{b}} + \sum_a\, \gamma_{\bar{b}a}\, \Big[\frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (4\, \partial_a\, \mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a^2}{\triangle}\, \sum_c\, \partial_c\, \mathcal{M}^{(UV)}_{(1)c}) + \frac{\partial_a^2}{\triangle}\, {}^3K_{(1)} \Big] \Big), \tag{4.38}$$

and this expression can be put in $\hat{E}_{ADM(2)}$ of Eq. (4.21). As a consequence, the effective Hamiltonian of the linearized theory in the 3-orthogonal gauges is (use Eq.(4.6) for $\tilde{\phi} = 1 + 6\,\phi_{(1)} + O(\zeta^2)$)

$$\begin{aligned}
H_{eff} &= M_{(1)}\,c + \frac{1}{c}\,\hat{E}_{ADM(2)} + \\
&+ \frac{c^3}{12\pi\,G}\int d^3\sigma\,\partial_\tau\,{}^3K_{(1)}(\tau,\vec{\sigma})\,\Big[1 + \frac{6}{\triangle}\Big(-\frac{2\pi\,G}{c^3}\,\mathcal{M}^{(UV)}_{(1)} + \frac{1}{4}\sum_b \partial_b^2\,\Gamma_b^{(1)}\Big)(\tau,\vec{\sigma})\Big] = \\
&= \int d^3\sigma\,\Big(\mathcal{M}^{(UV)}_{(1)} + \mathcal{M}^{(UV)}_{(2)} + \\
&+ \frac{8\pi\,G}{c^3}\,\Big[\frac{1}{4}\,\mathcal{M}^{(UV)}_{(1)}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} - \sum_a \mathcal{M}^{(UV)}_{(1)a}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)a} - \frac{1}{4}\Big(\sum_a \frac{\partial_a}{\triangle}\,\mathcal{M}^{(UV)}_{(1)a}\Big)^2\Big] + \\
&+ \Big(\sum_a \frac{\partial_a}{\triangle}\,\mathcal{M}^{(UV)}_{(1)a}\Big)\,{}^3K_{(1)} + \frac{c^3}{16\pi G}\sum_{\bar{a}\bar{b}}\sum_a \partial_a\,R_{\bar{a}}\,M_{\bar{a}\bar{b}}\,\partial_a\,R_{\bar{b}} + \\
&+ \frac{c^3}{16\pi G}\sum_{\bar{a}\bar{b}}\Big(\frac{8\pi\,G}{c^3}\,\Pi_{\bar{a}} + \sum_a \gamma_{\bar{a}a}\,\Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a^2}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}) + \frac{\partial_a^2}{\triangle}\,{}^3K_{(1)}\Big]\Big) \\
&\quad M^{-1}_{\bar{a}\bar{b}}\,\Big(\frac{8\pi\,G}{c^3}\,\Pi_{\bar{b}} + \sum_b \gamma_{\bar{b}b}\,\Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_b\,\mathcal{M}^{(UV)}_{(1)b} - \frac{\partial_b^2}{\triangle}\sum_d \partial_d\,\mathcal{M}^{(UV)}_{(1)d}) + \frac{\partial_b^2}{\triangle}\,{}^3K_{(1)}\Big]\Big) + \\
&+ \frac{c^3}{12\pi\,G}\,\partial_\tau\,{}^3K_{(1)}\,\Big[1 + \frac{6}{\triangle}\Big(-\frac{2\pi\,G}{c^3}\,\mathcal{M}^{(UV)}_{(1)} + \frac{1}{4}\sum_b \partial_b^2\,\Gamma_b^{(1)}\Big)(\tau,\vec{\sigma})\Big]\Big)(\tau,\vec{\sigma}).
\end{aligned} \tag{4.39}$$

As expected we get $\frac{d}{d\tau}\Big(M_{(1)}\,c^2 + \hat{E}_{ADM(2)}\Big) = \frac{\partial}{\partial\tau}\Big(M_{(1)}\,c^2 + \hat{E}_{ADM(2)}\Big) + \{M_{(1)}\,c^2 + \hat{E}_{ADM(2)}, H_{eff}\} = 0$ ($\frac{\partial}{\partial\tau}$ acts only on ${}^3K_{(1)}(\tau,\vec{\sigma})$). One could check that the Hamilton equations in the 3-orthogonal gauges are generated by this effective Hamiltonian

In conclusion, when ${}^3K_{(1)}(\tau,\vec{\sigma}) \approx F_{(1)}(\tau,\vec{\sigma})$ is τ-dependent the effective Hamiltonian is not the energy and there are additional inertial effects like it happens in non-inertial frames in Minkowski space-time [4].

V. THE LINEARIZED EQUATIONS FOR THE PARTICLES AND THE ELECTRO-MAGNETIC FIELD

A. The Equations of Motion of the Particles

Let us now study the weak field approximation of the Hamilton equations (2.18) for the particles. Since in this approximation we have

$$
\begin{aligned}
&\frac{1}{\sqrt{(1+n)^2 - \phi^4\, \sum_c Q_c^2\, (\dot{\eta}^c_i(\tau) + \phi^{-2}\, Q_c^{-1}\, \bar{n}_{(c)})^2}}(\tau, \vec{\eta}_i(\tau)) \; = \\
&= \frac{1}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}} \Big[1 - \frac{n_{(1)} - \sum_c \dot{\eta}^c_i(\tau)\, [\bar{n}_{(1)(c)} + (\Gamma_c^{(1)} + 2\, \phi_{(1)})\, \dot{\eta}^c_i(\tau)]}{1 - \dot{\vec{\eta}}^2_i(\tau)}\Big](\tau, \vec{\eta}_i(\tau)) + O(\zeta^2),
\end{aligned}
\tag{5.1}
$$

we get that the particle momenta (2.17) have the following expression in terms of the particle velocities

$$
\begin{aligned}
\frac{\kappa_{ir}(\tau)}{m_i\, c} &\circeq \frac{Q_i}{m_i\, c^2}\, A_{\perp\, r}(\tau, \vec{\eta}_i(\tau)) + \frac{1}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}} \Big[\dot{\eta}^r_i(\tau) \Big(1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)}) - \\
&- \frac{n_{(1)} - \sum_c \dot{\eta}^c_i(\tau)\, [\bar{n}_{(1)(c)} + (\Gamma_c^{(1)} + 2\, \phi_{(1)})\, \dot{\eta}^c_i(\tau)]}{1 - \dot{\vec{\eta}}^2_i(\tau)}\Big) + \bar{n}_{(1)(r)}\Big](\tau, \vec{\eta}_i(\tau)) = \\
&= \frac{\dot{\eta}^r_i(\tau)}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}} + \frac{M}{m_i}\, O(\zeta) + \frac{M}{m_i}\, O(\zeta^2), \\
&\quad or
\end{aligned}
$$

$$
\begin{aligned}
\frac{\kappa_{ir}(\tau)}{m_i\, c} &\circeq \frac{Q_i}{m_i\, c^2}\, A_{\perp\, r}(\tau, \vec{\eta}_i(\tau)) + \\
&+ \frac{1}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}} \Big[\dot{\eta}^r_i(\tau) \Big(1 - \frac{8\pi\, G}{c^3}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} + 2\, \Gamma_r^{(1)} + \sum_b \frac{\partial_b^2}{\triangle}\, \Gamma_b^{(1)} - \\
&- \frac{1}{1 - \dot{\vec{\eta}}^2_i(\tau)} \Big[\frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (\mathcal{M}^{(UV)}_{(1)} + \sum_b T^{bb}_{(1)}) - \\
&- \sum_c \dot{\eta}^c_i(\tau) \Big(\frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (4\, \mathcal{M}^{(UV)}_{(1)c} - \frac{\partial_c}{\triangle} \sum_b \partial_b\, \mathcal{M}^{(UV)}_{(1)b}) + \\
&+ \frac{1}{2}\, \partial_\tau\, \frac{\partial_c}{\triangle}\, (4\, \Gamma_c^{(1)} - \sum_b \frac{\partial_b^2}{\triangle}\, \Gamma_b^{(1)}\Big) +
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_c (\dot{\eta}_i^c(\tau))^2 \Big(- \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} + \Gamma_c^{(1)} + \frac{1}{2} \sum_b \frac{\partial_b^2}{\triangle}\, \Gamma_b^{(1)} \Big) \Big] \Big) + \\
&+ \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (4\, \mathcal{M}^{(UV)}_{(1)r} - \frac{\partial_r}{\triangle} \sum_b \partial_b\, \mathcal{M}^{(UV)}_{(1)b}) + \frac{1}{2}\, \partial_\tau\, \frac{\partial_r}{\triangle}\, (4\, \Gamma_r^{(1)} - \sum_b \frac{\partial_b^2}{\triangle}\, \Gamma_b^{(1)}) + \\
&+ \frac{1}{\triangle} \Big(\partial_r\, {}^3K_{(1)} + \frac{\dot{\eta}_i^r(\tau)}{1 - \dot{\vec{\eta}}_i^2(\tau)}\, \partial_\tau\, {}^3K_{(1)} \Big) \Big] (\tau, \vec{\eta}_i(\tau)),
\end{aligned}
\tag{5.2}
$$

where Eqs. (4.6), (4.7) and (4.16) were used to find the final expression. This result is implied by Eqs.(3.11): the ultraviolet cutoff implies a small deviation from the expression in the free case.

In the last expression the mass density $\mathcal{M}^{(UV)}_{(1)}$, the mass current density $\mathcal{M}^{(UV)}_{(1)r}$ and the tidal variables $\Gamma_a^{(1)}$, which will be shown to depend on the stress tensor $T^{rs}_{(1)}$ in Sections VI and VII, have to be evaluated by using the lowest order expression $\kappa_{ir}(\tau) - \frac{Q_i}{c}\, A_{\perp r}(\tau, \vec{\eta}_i(\tau)) \mapsto \frac{m_i\, c\, \dot{\eta}_i^r(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_i^2(\tau)}}$ (so that $\sqrt{m_i^2 c^2 + \Big(\vec{\kappa}_i(\tau) - \frac{Q_i}{c}\, \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) \Big)^2} \mapsto \frac{m_i\, c}{\sqrt{1 - \dot{\vec{\eta}}_i(\tau)}}$).

The particle momenta depend on the τ- and spatial-derivatives of the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$. If the York time ${}^3K_{(1)}$ would depend also on the particle positions, we should make the following replacement $\partial_\tau\, {}^3K_{(1)} \mapsto \partial_\tau\, {}^3K_{(1)}|_{\vec{\eta}_i} + \sum_b \dot{\eta}_i^b(\tau)\, \partial_b\, {}^3K_{(1)}$.

As a consequence, we can get the following second order form of the equations of motion (2.17) for the particles, implied by the Hamilton equations (we use the solutions (4.6), (4.7), (4.16) for $\phi_{(1)}$, $n_{(1)}$, $\bar{n}_{(1)(r)}$),

$$
\begin{aligned}
&\eta_i\, \frac{d}{d\tau} \Big[\frac{\dot{\eta}_i^r(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_i^2(\tau)}} \Big(1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)}) - \\
&- \frac{n_{(1)} - \sum_c \dot{\eta}_i^c(\tau)\, [\bar{n}_{(1)(c)} + (\Gamma_c^{(1)} + 2\, \phi_{(1)})\, \dot{\eta}_i^c(\tau)]}{1 - \dot{\vec{\eta}}_i^2(\tau)} \Big) + \\
&+ \frac{\bar{n}_{(1)(r)}}{\sqrt{1 - \dot{\vec{\eta}}_i^2(\tau)}} \Big] (\tau, \vec{\eta}_i(\tau)) \,\circeq
\end{aligned}
$$

$$
\begin{aligned}
&\stackrel{\circ}{=}\ \eta_i\,\frac{d}{d\tau}\Big(\frac{\dot{\eta}_i^r(\tau)}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}\Big[1+2\,\Gamma_r^{(1)}-\frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}+\sum_c\frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}-\\
&-\ \frac{1}{1-\dot{\vec{\eta}}_i^2(\tau)}\Big(\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(\mathcal{M}^{(UV)}_{(1)}+\sum_a T^{aa}_{(1)})-\frac{\partial_\tau}{\triangle}\,{}^3K_{(1)}-\\
&-\ \sum_d\dot{\eta}_i^d(\tau)\Big[\frac{\partial_d}{\triangle}\,{}^3K_{(1)}+\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\mathcal{M}^{(UV)}_{(1)d}-\frac{\partial_d}{\triangle}\sum_c\partial_c\,\mathcal{M}^{(UV)}_{(1)c})+\\
&+\ \frac{1}{2}\,\frac{\partial_d}{\triangle}\,\partial_\tau\,(4\,\Gamma_d^{(1)}-\sum_c\frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)})+\dot{\eta}_i^d(\tau)\,\Big(\Gamma_d^{(1)}-\frac{6\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}+\frac{1}{2}\sum_c\frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)\Big]\Big)\Big]+\\
&+\ \frac{1}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}\Big[\frac{\partial_r}{\triangle}\,{}^3K_{(1)}+\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\mathcal{M}^{(UV)}_{(1)r}-\frac{\partial_r}{\triangle}\sum_c\partial_c\,\mathcal{M}^{(UV)}_{(1)c})+\\
&+\ \frac{1}{2}\,\frac{\partial_r}{\triangle}\,\partial_\tau\,(4\,\Gamma_r^{(1)}-\sum_c\frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)})\Big](\tau,\vec{\eta}_i(\tau))\stackrel{\circ}{=}\\
&\quad\stackrel{\circ}{=}\frac{\eta_i\,Q_i}{m_i\,c^2}\,F_{rs}(\tau,\vec{\eta}_i(\tau))\,\dot{\eta}_i^s(\tau)-\frac{\eta_i}{m_i\,c}\,\frac{\partial\,\mathcal{W}[\tau,\vec{\eta}_i(\tau)]}{\partial\,\eta_i^r}+\frac{\eta_i}{m_i\,c}\,\check{F}_{ir}(\tau,\vec{\eta}_i(\tau)),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}[\tau,\vec{\eta}_i(\tau)]\ &=\ \int d^3\sigma\,\sum_a\Big[\Big(1+n_{(1)}+2\,(\Gamma_a^{(1)}-\phi_{(1)})\Big)\,\mathcal{W}_{(n)a}+\\
&+\ \Big(1-(\Gamma_a^{(1)}+2\,\phi_{(1)})\Big)\,\mathcal{W}_a\Big](\tau,\vec{\sigma})=\\
&=\ \int d^3\sigma\,\sum_a\Big[\Big(1+\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(2\,\mathcal{M}^{(UV)}_{(1)}+\sum_b T^{bb}_{(1)})+2\,\Gamma_a^{(1)}-\\
&-\ \frac{1}{2}\sum_b\frac{\partial_b^2}{\triangle}\,\Gamma_b^{(1)}-\frac{1}{\triangle}\,\partial_\tau\,{}^3K_{(1)}\Big)\,\mathcal{W}_{(n)a}+\\
&+\ \Big(1+\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}-\Gamma_a^{(1)}-\frac{1}{2}\sum_b\frac{\partial_b^2}{\triangle}\,\Gamma_b^{(1)}\Big)\,\mathcal{W}_a\Big](\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{W}_{(n)a}(\tau,\vec{\sigma})=-\frac{1}{2c}\,\Big(2\,\pi_\perp^a(\tau,\vec{\sigma})-\sum_{j\neq i,k}\eta_j\,Q_j\,\partial_a\,c(\vec{\sigma},\vec{\eta}_j(\tau))\Big)\,\sum_{k\neq i}\eta_k\,Q_k\,\partial_a\,c(\vec{\sigma},\vec{\eta}_k(\tau)),\\
&\mathcal{W}_a(\tau,\vec{\sigma})=-\frac{1}{c}\sum_s F_{as}(\tau,\vec{\sigma})\,\sum_{k\neq i}\eta_k\,Q_k\,\partial_s\,c(\vec{\sigma},\vec{\eta}_k(\tau)),
\end{aligned}
$$

$$
\begin{aligned}
\check{F}_{ir}(\tau, \vec{\eta}_i(\tau)) \;&=\; \frac{m_i\, c}{\sqrt{1 - \dot{\vec{\eta}}_i^{\,2}(\tau)}} \Big[\sum_a \dot{\eta}_i^a(\tau) \Big(\frac{\partial\, \bar{n}_{(1)(a)}}{\partial\, \eta_i^r} + \frac{\partial\, (\Gamma_a^{(1)} + 2\, \phi_{(1)})}{\partial\, \eta_i^r}\, \dot{\eta}_i^a(\tau) \Big) - \\
&-\; \frac{\partial\, n_{(1)}}{\partial\, \eta_i^r} \Big] (\tau, \vec{\eta}_i(\tau)) = \\
&=\; \frac{m_i\, c}{\sqrt{1 - \dot{\vec{\eta}}_i^{\,2}(\tau)}}\, \frac{\partial}{\partial \eta_i^r} \Big[\sum_a \dot{\eta}_i^a(\tau) \Big(\frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (4\, \mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a}{\triangle} \sum_c \partial_c\, \mathcal{M}^{(UV)}_{(1)c}) + \\
&+\; \frac{1}{2}\, \partial_\tau\, \frac{\partial_a}{\triangle}\, (4\, \Gamma_a^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma_c^{(1)}) + \frac{\partial_a}{\triangle}\, {}^3K_{(1)} + \\
&+\; \dot{\eta}_i^a(\tau) \Big[- \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} + \Gamma_a^{(1)} + \frac{1}{2} \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma_c^{(1)} \Big] \Big) - \\
&-\; \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, (\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}) - \frac{1}{\triangle}\, \partial_\tau\, {}^3K_{(1)} \Big] (\tau, \vec{\eta}_i(\tau)),
\end{aligned}
\tag{5.3}
$$

where $\mathcal{W}$ is the non-inertial Coulomb potential term and $\check{F}_{ir}$ are generalized inertial forces (now function of the inertial and tidal components of the gravitational field). As a consequence, the deviation from free motions is at the first order, consistently with the weak field approximation.

Let us remark that in absence of the electro-magnetic field the final form of the equations of motion of particle i does not depend upon the mass m_i like it happens for test particles following geodesics. Therefore the masses m_i are playing both the role of inertial and gravitational mass of the particles: their equality implies that the final form of the equations for particle i only depends on the masses $m_{j \neq i}$ present in the mass density, in the mass current density and in the tidal variables.

These equations of motion for dynamical (not test) scalar particles with a definite sign of the energy (implied by the Grassmann charges η_i as shown in paper I) are not thought as the point limit of small extended objects as in Ref.[16] and do not contain terms corresponding to a gravitational self-force [16, 17] because $\eta_i^2\, m_i^2 = 0$ (only terms $\eta_i\, \eta_j\, m_i\, m_j$ with $i \neq j$ appear). As shown in Section VII with this description we can still get the energy balance when GW's are emitted.

The Newtonian limit of the equations of motion for the particle in absence of the electro-magnetic field will be studied in the third paper [18].

B. The Equations of Motion of the Transverse Electro-Magnetic Field

The weak field limit of Eqs.(2.19) is

$$\partial_\tau\, A_{\perp r}(\tau,\vec{\sigma}) \,\stackrel{\circ}{=}\, \sum_{na} \delta_{rn}\, P_{\perp}^{na}(\vec{\sigma}) \Big([1 + n_{(1)} - 2\,(\Gamma_a^{(1)} - \phi_{(1)})]\, \pi_{\perp}^a + \\ + \sum_b \bar{n}_{(1)(b)}\, F_{ba}\Big)(\tau,\vec{\sigma}) = \sum_n \delta_{rs}\, \pi_{\perp}^s(\tau,\vec{\sigma}) + O(\zeta),$$

$$\partial_\tau\, \pi_{\perp}^r(\tau,\vec{\sigma}) \,\stackrel{\circ}{=}\, \sum_a P_{\perp}^{ra}(\vec{\sigma}) \Big(\sum_i \eta_i\, Q_i\, \delta^3(\vec{\sigma},\vec{\eta}_i(\tau)) \Big[\frac{\kappa_{ia}(\tau)}{\sqrt{m_i^2 c^2 + \sum_b \Big(\kappa_{ib}(\tau) - \frac{Q_i}{c}\, A_{\perp b}\Big)^2}} \Big(1 + \\ + \, n_{(1)} - 2\,(\Gamma_a^{(1)} + 2\,\phi_{(1)}) + \frac{\sum_b (\Gamma_b^{(1)} + 2\,\phi_{(1)}) \Big(\kappa_{ib}(\tau) - \frac{Q_i}{c}\, A_{\perp b}\Big)^2}{m_i^2 c^2 + \sum_b \Big(\kappa_{ib}(\tau) - \frac{Q_i}{c}\, A_{\perp b}\Big)^2} \Big) - \bar{n}_{(1)(a)} \Big](\tau,\vec{\eta}_i(\tau)) - \\ - \, \Big[\sum_b \Big([1 + n_{(1)} - 2\,(\Gamma_a^{(1)} + \Gamma_b^{(1)} + \phi_{(1)})]\, \partial_b\, F_{ab} - \\ - \, [\partial_b\, \phi_{(1)} - \partial_b\, n_{(1)} + \partial_b\, (\Gamma_a^{(1)} + \Gamma_b^{(1)})]\, F_{ab}\Big) - \sum_b \bar{n}_{(1)(b)}\, \partial_b\, \pi_{\perp}^a + \\ + \, \sum_i \eta_i\, Q_i \sum_b \Big(\partial_b\, \bar{n}_{(1)(b)}\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\, \sigma^a} - \partial_b\, \bar{n}_{(1)(a)}\, \frac{\partial\, c(\vec{\sigma},\vec{\eta}_i(\tau))}{\partial\, \sigma^b} \Big) \Big](\tau,\vec{\sigma}) \Big). \qquad (5.4)$$

The explicit expression of these equations in terms of the instantaneous inertial and retarded gravitational quantities can be obtained by using Eqs. (4.6), (4.7) and (4.16). At the lowest order and using Eq.(5.2) these equations imply the special relativistic wave equation $\Box\, A_{\perp r}(\tau,\vec{\sigma}) \stackrel{\circ}{=} \sum_{nsm} \delta_{rn}\, P_{\perp}^{ns}(\vec{\sigma})\, \delta_{sm} \sum_i \eta_i\, Q_i\, \dot{\eta}_i^m(\tau)\, \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))$, whose Lienard-Wiechert solution was found and put in Hamiltonian form in Ref.[15].

The first of Eqs.(5.4) can be inverted by iteration and the resulting form of the transverse electro-magnetic momenta is

$$\pi_{\perp}^r(\tau,\vec{\sigma}) \,\stackrel{\circ}{=}\, \sum_a P_{\perp}^{ra}(\vec{\sigma}) \Big([1 - n_{(1)} + 2\,(\Gamma_a^{(1)} - \phi_{(1)})]\, \partial_\tau\, A_{\perp\, a} - \\ - \sum_b \bar{n}_{(1)(b)}\, F_{ba}\Big)(\tau,\vec{\sigma}) + O(\zeta^2). \qquad (5.5)$$

If we put Eq.(5.5) into the second of Eqs.(5.4) we get a wave equation for $A_{\perp\, r}(\tau,\vec{\sigma})$ containing extra terms in $\partial_\tau\, A_{\perp\, a}$ and $\partial_b\, \partial_\tau\, A_{\perp\, a}$, which will be studied elsewhere. By using the solutions $\phi_{(1)}$, $n_{(1)}$, $\bar{n}_{(1)(a)}$, we see that this equation has as sources both the matter and the tidal variables. Since in the next two Sections we will show that the tidal variables $\Gamma_r^{(1)}$ are determined by the stress tensor $T_{(1)}^{rs}$ in a retarded way, the final equations for the transverse electro-magnetic field depend on the matter both in an (action-at-a-distance) instantaneous way and in a retarded way. If one could find a Lienard-Wiechert-type solution of these equations, the final form of the particle equations (5.3) would be of coupled integro-differential equations instead of differential equations like it happens in absence of the gravitational field.

VI. THE LINEARIZED SECOND ORDER EQUATIONS FOR THE TIDAL VARIABLES $R_{\bar a}$ IN THE 3-ORTHOGONAL GAUGES

In Section IV we solved the equations of elliptic type for $\phi_{(1)}$, $n_{(1)}$, $\bar n_{(1)(r)}$ and $\sigma_{(1)(a)(b)}|_{a\not= b}$. The solutions depend on the tidal variables $R_{\bar a}$.

We must now study the linearization of the second order equations (2.16) for the tidal variables by using the solutions of Section IV. We will see that also in these non-harmonic 3-orthogonal gauges we get wave equations, but they will also contain the information that the final 3-metric is traceless and transverse (TT).

A. The Linearization of Eqs.(2.16).

For the three integrals appearing in Eqs.(2.16) we get the following linearization:

a) the last integral in Eq.(2.16)) becomes

$$\begin{aligned}
&\int d^3\sigma_1\, \Big(1 + n(\tau, \vec \sigma_1)\Big)\, {{\delta\, {\cal M}(\tau, \vec \sigma_1)}\over {\delta\, R_{\bar a}(\tau, \vec \sigma)}} \, =\\
&=\, - \sum_i\, \delta^3(\vec \sigma, {\vec \eta}_i(\tau))\, \eta_i\, \Big({{\sum_a\, \gamma_{\bar aa}\, \Big(\kappa_{ia}(\tau) - {{Q_i}\over c}\, A_{\perp\, a}\Big)^2}\over {\sqrt{m_i^2\, c^2 + \sum_a\, \Big(\kappa_{ia}(\tau) - {{Q_i}\over c}\, A_{\perp\, a}\Big)^2}}}\Big)(\tau, \vec \sigma) +\\
&+ {1\over c}\, \Big[\sum_a\, \gamma_{\bar aa}\, (\pi^a_{\perp})^2 - \sum_{ab}\, \gamma_{\bar aa}\, F^2_{ab} - \sum_a\, \gamma_{\bar aa} \Big(2\, \pi^a_{\perp} - \sum_{k\not= j}\, \eta_k\, Q_k\, \partial_a\, c(\vec \sigma, {\vec \eta}_k(\tau))\Big)\\
&\sum_j\, \eta_j\, Q_j\, \partial_a\, c(\vec \sigma, {\vec \eta}_j(\tau))\Big](\tau, \vec \sigma) + O(mc\, \zeta^2) =\\
=&\, - \sum_a\, \gamma_{\bar aa}\, T^{aa}_{(1)} + O(\zeta^2),\qquad from\,\, Eq.(3.12).
\end{aligned}\tag{6.1}$$

b) the first integral in Eq.(2.16) becomes

$$\begin{aligned}
&\int d^3\sigma_1\, \, [1 + n(\tau, \vec \sigma_1)]\, {{\delta\, {\cal S}(\tau, \vec \sigma_1)}\over {\delta\, R_{\bar a}(\tau, \vec \sigma)}}|_{\theta^i = 0}\, =\\
&=\, 4\, \sum_a\, \gamma_{\bar aa}\, \partial_a^2\, \phi_{(1)}(\tau, \vec \sigma) - 2\, \sum_{a\bar b}\, (2\, \gamma_{\bar aa}\, \gamma_{\bar ba} - \delta_{\bar a\bar b})\, \partial^2_a\, R_{\bar b}(\tau, \vec \sigma) + O(\zeta^2).
\end{aligned}\tag{6.2}$$

c) the second integral in Eq.(2.16) becomes

$$\int d^3\sigma_1\, n(\tau, \vec \sigma_1)\, {{\delta\, {\cal T}(\tau, \vec \sigma_1)}\over {\delta\, R_{\bar a}(\tau, \vec \sigma)}}|_{\theta^i = 0}\, =\, 2\, \sum_a\, \gamma_{\bar aa}\, \partial^2_a\, n_{(1)}(\tau, \vec \sigma) + O(\zeta^2),\tag{6.3}$$

As a consequence, the linearization of the second order equations (2.16) for the tidal variables $R_{\bar a}$ is

$$\begin{aligned}
\partial_\tau^2\, R_{\bar a}(\tau, \vec\sigma) &\,\stackrel{\circ}{=}\, \triangle\, R_{\bar a}(\tau, \vec\sigma) + \sum_a \gamma_{\bar a a} \Big[\partial_\tau\, \partial_a\, \bar n_{(1)(a)} + \\
&+\; \partial_a^2\, n_{(1)} + 2\, \partial_a^2\, \phi_{(1)} - 2\, \partial_a^2\, \Gamma_a^{(1)} + \frac{8\pi\, G}{c^3}\, T^{aa}_{(1)}\Big](\tau, \vec\sigma), \\
&\qquad \Downarrow \qquad by\, using\, Eqs.(4.6),\, (4.7),\, (4.16)
\end{aligned}$$

$$\begin{aligned}
\Box \sum_{\bar b} M_{\bar a\bar b}\, R_{\bar b} &\,\stackrel{\circ}{=}\, E_{\bar a}, \\
& M_{\bar a\bar b} = \delta_{\bar a\bar b} - \sum_a \gamma_{\bar a a}\, \frac{\partial_a^2}{\triangle} \Big(2\, \gamma_{\bar b a} - \frac{1}{2} \sum_b \gamma_{\bar b b}\, \frac{\partial_b^2}{\triangle}\Big), \\
& E_{\bar a} = \frac{4\pi\, G}{c^3} \sum_a \gamma_{\bar a a} \Big[\frac{\partial_\tau\, \partial_a}{\triangle} \Big(4\, \check{\mathcal{M}}^{(UV)}_{(1)a} - \frac{\partial_a}{\triangle} \sum_c \partial_c\, \check{\mathcal{M}}^{(UV)}_{(1)c}\Big) + \\
&\quad +2\, T^{aa}_{(1)} + \frac{\partial_a^2}{\triangle} \sum_b T^{bb}_{(1)}\Big], \\
& \Downarrow \\
\Box \sum_b \tilde M_{ab}\, \Gamma_b^{(1)} &\,\stackrel{\circ}{=}\, \sum_{\bar a} \gamma_{\bar a a}\, E_{\bar a}, \\
& \tilde M_{ab} = \sum_{\bar a\bar b} \gamma_{\bar a a}\, \gamma_{\bar b b}\, M_{\bar a\bar b} = \delta_{ab} \Big(1 - 2\, \frac{\partial_a^2}{\triangle}\Big) + \frac{1}{2} \Big(1 + \frac{\partial_a^2}{\triangle}\Big)\, \frac{\partial_b^2}{\triangle}, \\
& \sum_a \tilde M_{ab} = 0, \qquad M_{\bar a\bar b} = \sum_{ab} \gamma_{\bar a a}\, \gamma_{\bar b b}\, \tilde M_{ab},
\end{aligned} \tag{6.4}$$

where we used Eqs.(4.6), (4.7), (4.16) and $\sum_{\bar a} \gamma_{\bar a a}\, \gamma_{\bar a b} = \delta_{ab} - \frac{1}{3}$, $\sum_a \gamma_{\bar a a}\, \gamma_{\bar b a} = \delta_{\bar a\bar b}$, $\sum_a \gamma_{\bar a a} = 0$.

Therefore we get the massless wave equation (with the flat d'Alambertian $\Box$ associated to the asymptotic Minkowski metric) not for the tidal variable $R_{\bar a}$ but for the quantity $\sum_{\bar b} M_{\bar a\bar b}\, R_{\bar b}$, where $M_{\bar a\bar b}$ is the spatial operator already found in Eq.(4.18).

B. The Meaning of the Operators $M_{\bar a\bar b}$ and $\tilde M_{ab}$

Let us show that the operators $M_{\bar a\bar b}$ and $\tilde M_{ab}$ are present to select the TT part of the spatial metric ${}^4g_{(1)rs} = -\epsilon\, {}^3g_{(1)rs} = -\epsilon\, \delta_{rs} + {}^4h_{(1)rs} = -\epsilon\, \delta_{rs} \Big(1 + 2(\Gamma_r^{(1)} + 2\, \phi_{(1)})\Big)$, namely that they define the polarization pattern of the gravitational waves in these non-harmonic 3-orthogonal gauges.

In Ref.[20] it is shown that in every gauge we can make the following decomposition of $h_{(1)rs}(\tau, \vec{\sigma})$

$$
{}^4h_{(1)rs} = {}^4h^{TT}_{(1)rs} + \frac{1}{3}\,\delta_{rs}\,H_{(1)} + \frac{1}{2}\,(\partial_r\,\epsilon_{(1)s} + \partial_s\,\epsilon_{(1)r}) + (\partial_r\,\partial_s - \frac{1}{3}\,\delta_{rs}\,\triangle)\,\lambda_{(1)}, \tag{6.5}
$$

with $\sum_r \partial_r\,\epsilon_{(1)r} = 0$ and ${}^4h^{TT}_{(1)rs}$ traceless and transverse, i.e. $\sum_r {}^4h^{TT}_{(1)rr} = 0$, $\sum_r \partial_r\,{}^4h^{TT}_{(1)rs} = 0$. The functions $H_{(1)}$, $\lambda_{(1)}$ and $\epsilon_{(1)r}$ have the following expression

$$
\begin{aligned}
H_{(1)} &= {}^4h_{(1)} = \sum_{rs} \delta^{rs}\,{}^4h_{(1)rs} = \sum_r {}^4h_{(1)rr} = -12\,\epsilon\,\phi_{(1)},\\
\lambda_{(1)} &= \frac{3}{2}\,\frac{1}{\triangle^2}\,\sum_{uv}\Big(\partial_u\,\partial_v\,-\frac{1}{3}\,\delta_{uv}\,\triangle\Big)\,{}^4h_{(1)uv} = -3\,\epsilon\,\sum_u \frac{\partial_u^2}{\triangle^2}\,\Gamma^{(1)}_u,\\
\epsilon_{(1)r} &= \frac{2}{\triangle}\,\Big(\sum_u \partial_u\,{}^4h_{(1)ur} - \frac{\partial_r}{\triangle}\,\sum_{uv}\partial_u\,\partial_v\,{}^4h_{(1)uv}\Big) =\\
&= -4\,\epsilon\,\frac{\partial_r}{\triangle}\,\Big(\Gamma^{(1)}_r - \sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big),
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
{}^4h_{(1)rs} &= {}^4h^{TT}_{(1)rs} - \epsilon\,\Big[\Big(4\,\phi_{(1)} - \sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big)\,\delta_{rs} +\\
&+ 2\,\frac{\partial_r\,\partial_s}{\triangle}\,\Big(\Gamma^{(1)}_r + \Gamma^{(1)}_s - \frac{1}{2}\,\sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big).
\end{aligned} \tag{6.6}
$$

In the last line of this equation we have given the form of the 3-metric implied by Eq.(6.5).

As a consequence the TT part of the spatial metric is independent from $\phi_{(1)}$ and has the expression

$$
\begin{aligned}
{}^4h^{TT}_{(1)rs} &= {}^4h_{(1)rs} + \frac{1}{2}\,\delta_{rs}\,\Big(\sum_{uv}\frac{\partial_u\,\partial_v}{\triangle}\,{}^4h_{(1)uv} - \sum_u {}^4h_{(1)uu}\Big) +\\
&+ \frac{1}{2}\,\frac{\partial_r\,\partial_s}{\triangle}\,\Big(\sum_{uv}\frac{\partial_u\,\partial_v}{\triangle}\,{}^4h_{(1)uv} + \sum_u {}^4h_{(1)uu}\Big) - \sum_u \frac{\partial_u}{\triangle}\,\Big(\partial_r\,{}^4h_{(1)us} + \partial_s\,{}^4h_{(1)ur}\Big) =\\
&= -\epsilon\,\Big[\Big(2\,\Gamma^{(1)}_r + \sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big)\,\delta_{rs} - 2\,\frac{\partial_r\,\partial_s}{\triangle}\,(\Gamma^{(1)}_r + \Gamma^{(1)}_s) + \frac{\partial_r\,\partial_s}{\triangle}\,\sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big] =\\
&= \sum_{uv}\mathcal{P}_{rsuv}\,{}^4h_{(1)uv},\\
\mathcal{P}_{rsuv} &= \frac{1}{2}\,(\delta_{ru}\,\delta_{sv} + \delta_{rv}\,\delta_{su}) - \frac{1}{2}\,\Big(\delta_{rs} - \frac{\partial_r\,\partial_s}{\triangle}\Big)\,\delta_{uv} + \frac{1}{2}\,\Big(\delta_{rs} + \frac{\partial_r\,\partial_s}{\triangle}\Big)\,\frac{\partial_u\,\partial_v}{\triangle} -\\
&- \frac{1}{2}\,\Big[\frac{\partial_u}{\triangle}\,(\delta_{rv}\,\partial_s + \delta_{sv}\,\partial_r) + \frac{\partial_v}{\triangle}\,(\delta_{ru}\,\partial_s + \delta_{su}\,\partial_r)\Big],
\end{aligned} \tag{6.7}
$$

where $\mathcal{P}_{rsuv}$ is the projector extracting the TT part of the spatial metric. It satisfies $\sum_{uv} \mathcal{P}_{rsuv}\, \mathcal{P}_{uvmn} = \mathcal{P}_{rsmn} = \mathcal{P}_{srmn} = \mathcal{P}_{rsnm}$, $\sum_r \partial_r\, \mathcal{P}_{rsuv} = \sum_u \partial_u\, \mathcal{P}_{rsuv} = 0$, $\sum_r \mathcal{P}_{rruv} = \sum_u \mathcal{P}_{rsuu} = 0$.

With plane waves, i.e. ${}^4h_{(1)rs}(\tau, \vec{\sigma}) = A_{rs}(\tau)\, e^{i\, \vec{n}\cdot\vec{\sigma}} + cc$, we recover the standard projector $\mathcal{P}_{rsuv} \mapsto \Lambda_{rsuv} = P_{ru}\, P_{sv} - \frac{1}{2}\, P_{rs}\, P_{uv}$ with $P_{rs} = \delta_{rs} - n_r\, n_s$ ($\vec{n}$ is a unit vector orthogonal to the wave-front).

The diagonal elements ${}^4h^{TT}_{(1)aa}$ of the TT 3-metric contain the operator $\tilde{M}_{ab}$ of Eq.(6.4) since we have

$$
\begin{aligned}
{}^4h^{TT}_{(1)aa} &= -2\, \epsilon\, \Big[\Big(1 - 2\, \frac{\partial_a^2}{\triangle}\Big)\, \Gamma^{(1)}_a + \frac{1}{2}\, \Big(1 + \frac{\partial_a^2}{\triangle}\Big)\, \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c\Big] = \\
&= -2\, \epsilon\, \sum_c \tilde{M}_{ac}\, \Gamma^{(1)}_c, \\
\Rightarrow \qquad & \Box\, {}^4h^{TT}_{(1)aa} \circeq -\, 2\, \epsilon\, \sum_{\bar{a}} \gamma_{\bar{a}a}\, E_{\bar{a}}.
\end{aligned}
\tag{6.8}
$$

If we apply the decomposition (6.5) to the stress tensor $T^{rs}_{(1)}$ of Eqs.(3.12), we get

$$
\begin{aligned}
T^{rs}_{(1)} &= \frac{1}{3}\, \tilde{H}_{(1)}\, \delta_{rs} + T^{(TT)}_{(1)rs} + \frac{1}{2}\, (\partial_r\, \tilde{\epsilon}_{(1)s} + \partial_s\, \tilde{\epsilon}_{(1)r}) + (\partial_r\, \partial_s - \frac{1}{3}\, \delta_{rs}\, \triangle)\, \tilde{\lambda}_{(1)}, \\
& \sum_r \partial_r\, \tilde{\epsilon}_{(1)r} = \sum_r \partial_r\, T^{(TT)rs}_{(1)} = \sum_r T^{(TT)rr}_{(1)} = 0, \\
\tilde{H}_{(1)} &= \sum_r T^{rr}_{(1)}, \\
\tilde{\lambda}_{(1)} &= \frac{3}{2}\, \frac{1}{\triangle^2}\, \sum_{rs} (\partial_r\, \partial_s - \frac{1}{3}\, \delta_{rs}\, \triangle)\, T^{rs}_{(1)}, \\
\tilde{\epsilon}_{(1)r} &= 2\, \sum_u (\partial_u\, T^{ru}_{(1)} - \frac{1}{3}\, \partial_r\, T^{uu}_{(1)}) - 2\, \frac{\partial_r}{\triangle}\, \sum_{uv} (\partial_u\, \partial_v - \frac{1}{3}\, \delta_{uv}\, \triangle)\, T^{uv}_{(1)}, \\
T^{(TT)ab}_{(1)} &= \sum_{cd} \mathcal{P}_{abcd}\, T^{cd}_{(1)}, \\
T^{(TT)aa}_{(1)} &= \sum_{cd} \Big[\delta_{ac}\, \delta_{ad} - 2\, \delta_{ad}\, \frac{\partial_a\, \partial_c}{\triangle} + \frac{1}{2}\, (1 + \frac{\partial_a^2}{\triangle})\, \frac{\partial_c\, \partial_d}{\triangle} - \frac{1}{2}\, \delta_{cd}\, (1 - \frac{\partial_a^2}{\triangle})\Big]\, T^{cd}_{(1)},
\end{aligned}
\tag{6.9}
$$

where $T^{(TT)ab}_{(1)}$ is the TT part of the stress tensor.

By using Eqs.(3.13), i.e. $\partial_\tau\, \mathcal{M}^{(UV)}_{(1)} \circeq -\sum_c \partial_c\, \mathcal{M}^{(UV)}_c + \partial_A\, \mathcal{R}^{A\tau}_{(2)}$, $\partial_\tau\, \mathcal{M}^{(UV)}_a \circeq -\sum_c \partial_c\, T^{ca}_{(1)} + \partial_A\, \mathcal{R}^{Aa}_{(2)}$, so that $\partial^2_\tau\, \mathcal{M}^{(UV)}_{(1)} \circeq \sum_{cd} \partial_c\, \partial_d\, T^{cd}_{(1)} + \partial_A\, (\partial_\tau\, \mathcal{R}^{A\tau}_{(2)} + \sum_c \partial_c\, \mathcal{R}^{Ac}_{(2)})$, we get the following expression for the source term appearing in the second member of Eq.(6.4)

$$\sum_{\bar{a}} \gamma_{\bar{a}a}\, E_{\bar{a}} \stackrel{\circ}{=} \frac{8\pi\, G}{c^3}\, T^{(TT)aa}_{(1)}, \qquad E_{\bar{a}} \stackrel{\circ}{=} \frac{8\pi\, G}{c^3} \sum_a \gamma_{\bar{a}a}\, T^{(TT)aa}_{(1)}. \tag{6.10}$$

Therefore we get that the TT part of the 3-metric satisfies the wave equation

$$\begin{aligned}
\Box\, {}^4h^{TT}_{(1)aa} &\stackrel{\circ}{=} -2\,\epsilon \sum_{\bar{a}} \gamma_{\bar{a}a}\, E_{\bar{a}} \stackrel{\circ}{=} -\,\epsilon\, \frac{16\pi\, G}{c^3}\, T^{(TT)aa}_{(1)},\\
&\Downarrow\\
\Box\, {}^4h^{TT}_{(1)ab} &\stackrel{\circ}{=} -\epsilon\, \frac{16\pi\, G}{c^3}\, T^{(TT)ab}_{(1)}.
\end{aligned} \tag{6.11}$$

The second line of Eqs.(6.11) derives from the first line due to the transversality property of ${}^4h^{TT}_{(1)rs}$ and $T^{(TT)}_{(1)rs}$, which implies $\sum_{b\not= a} \partial_b \Big(\Box\, {}^4h^{TT}_{(1)ba} + \epsilon\, \frac{16\pi\, G}{c^3}\, T^{(TT)}_{(1)ba}\Big) = -\partial_a \Big(\Box\, {}^4h^{TT}_{(1)aa} + \epsilon\, \frac{16\pi\, G}{c^3}\, T^{(TT)}_{(1)aa}\Big) \stackrel{\circ}{=} 0$.

Therefore, even if we are not in a harmonic gauge, the Hamilton equations in the 3-orthogonal gauges imply *the massless wave equation for the TT part of the spatial metric*.

Once Eq.(6.11) is solved for ${}^4h^{(TT)}_{(1)aa} = -2\,\epsilon\, \sum_b \tilde{M}_{ab}\, \Gamma_b^{(1)}$, see Eq.(6.8), in terms of $T^{(TT)aa}_{(1)}$, we can find the solution for $R_{\bar{a}} = \sum_a \gamma_{\bar{a}a}\, \Gamma_a^{(1)}$ if we succeed to invert the operator $\tilde{M}_{ab}$. Then, if we put this solution in the last line of Eqs. (6.6), we can find the spatial metric ${}^4h_{(1)rs}$ in the 3-orthogonal gauges.

C. A Generalized TT Gauge

At this stage the 4-metric of Eqs.(3.2) and (3.5) has the following form

$$\begin{aligned}
{}^4g_{(1)AB} &= {}^4\eta_{AB} + \epsilon \begin{pmatrix} 2\, n_{(1)} & -\bar{n}_{(1)(r)} \\ -\bar{n}_{(1)(s)} & -2\, \Big(\Gamma_r^{(1)} + 2\, \phi_{(1)}\Big)\, \delta_{rs} \end{pmatrix} + O(\zeta^2) =\\
&= {}^4\eta_{AB} + \epsilon \begin{pmatrix} -2\, \frac{\partial_\tau}{\Delta}\, {}^3K_{(1)} + \alpha(matter) & -\frac{\partial_r}{\Delta}\, {}^3K_{(1)} + A_r(\Gamma_a^{(1)}) + \beta_r(matter) \\ -\frac{\partial_s}{\Delta}\, {}^3K_{(1)} + A_s(\Gamma_a^{(1)}) + \beta_s(matter) & \Big[B_r(\Gamma_a^{(1)}) + \gamma(matter)\Big]\, \delta_{rs} \end{pmatrix} +\\
&+\ O(\zeta^2),
\end{aligned}$$

$$
\begin{aligned}
A_r(\Gamma_a^{(1)}) &= -\frac{1}{2}\,\partial_\tau\,\frac{\partial_r}{\triangle}\Big(4\,\Gamma_r^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big),\\
B_r(\Gamma_a^{(1)}) &= -2\,\Big(\Gamma_r^{(1)} + \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big),\\
\alpha(matter) &= \frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(\mathcal{M}_{(1)}^{(UV)} + \sum_c T_{(1)}^{cc}\Big),\\
\beta_r(matter) &= -\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(4\,\mathcal{M}_{(1)r}^{(UV)} - \frac{\partial_r}{\triangle}\sum_c \partial_c\,\mathcal{M}_{(1)c}^{(UV)}\Big),\\
\gamma(matter) &= \frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}_{(1)}^{(UV)},
\end{aligned}
\tag{6.12}
$$

by using the expression of $\phi_{(1)}$, $n_{(1)}$ and $\bar{n}_{(1)(r)}$ given in Eqs. (4.6), (4.7) and (4.16), respectively. We have explicitly shown the dependence upon the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$, the instantaneous inertial effects $\alpha(matter)$, $\beta_r(matter)$, $\gamma(matter)$ and the retarded tidal effects $A_r(\Gamma_a^{(1)})$, $B_r(\Gamma_a^{(1)})$.

Let us consider the following coordinate transformation on the 3-space Σ_τ (endorsing it with τ-dependent new radar 3-coordinates)

$$
\begin{aligned}
&\bar{\tau} = \tau, \quad \bar{\sigma}^r = \sigma^r - \Psi_{(1)}^r(\tau, \vec{\sigma}),\\
&\qquad \Psi_{(1)}^r(\tau, \vec{\sigma}) = -\frac{1}{2}\,\frac{\partial_r}{\triangle}\Big(4\,\Gamma_r^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)(\tau, \vec{\sigma}) = O(\zeta),
\end{aligned}
\tag{6.13}
$$

and the associated new 4-metric ${}^4\bar{g}_{(1)AB}(\tau, \vec{\bar{\sigma}}) = \frac{\partial\,\sigma^C}{\partial\,\bar{\sigma}^A}\,\frac{\partial\,\sigma^D}{\partial\,\bar{\sigma}^B}\,{}^4g_{(1)CD}(\tau, \vec{\sigma})$ replacing the one of Eq.(6.12). The inverse transformation is $\tau = \bar{\tau}$, $\sigma^r = \bar{\sigma}^r + \Psi_{(1)}(\tau, \vec{\bar{\sigma}}) + O(\zeta^2)$.

Since we have $\frac{\partial\,\bar{\sigma}^r}{\partial\,\sigma^s} = \delta^r_s - \frac{\partial\,\Psi_{(1)}^r(\tau,\vec{\sigma})}{\partial\,\sigma^s}$, $\frac{\partial\,\bar{\sigma}^r}{\partial\,\tau} = -\frac{\partial\,\Psi_{(1)}^r(\tau,\vec{\sigma})}{\partial\,\tau}$, we get $\frac{\partial\,\sigma^r}{\partial\,\bar{\sigma}^s} = \delta^r_s + \frac{\partial\,\Psi_{(1)}^r(\tau,\vec{\bar{\sigma}})}{\partial\,\bar{\sigma}^s} + O(\zeta^2)$, $\frac{\partial\,\sigma^r}{\partial\,\bar{\tau}} = \frac{\partial\,\Psi_{(1)}^r(\tau,\vec{\bar{\sigma}})}{\partial\,\tau} + O(\zeta^2)$. In particular, by using Eqs.(4.6), (4.7) and (4.16), we get (since $\bar{\tau} = \tau$ and $\vec{\bar{\sigma}}$, we always use τ)

$$
\begin{aligned}
{}^4\bar{g}_{(1)\tau\tau}(\tau, \vec{\bar{\sigma}}) &= {}^4g_{(1)\tau\tau}(\tau, \vec{\sigma}) + O(\zeta^2) = \epsilon\,\Big[1 + 2\,n_{(1)}\Big](\tau, \vec{\bar{\sigma}}) + O(\zeta^2) =\\
&= \epsilon\,\Big[-2\,\frac{\partial_\tau}{\triangle}\,{}^3K_{(1)} + \frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(\mathcal{M}_{(1)}^{(UV)} + \sum_c T_{(1)}^{cc}\Big)\Big](\tau, \vec{\bar{\sigma}}) + O(\zeta^2),\\
{}^4\bar{g}_{(1)\tau r}(\tau, \vec{\bar{\sigma}}) &= {}^4g_{(1)\tau r}(\tau, \vec{\sigma}) - \epsilon\,\frac{\partial\,\Psi_{(1)}(\tau, \vec{\bar{\sigma}})^r}{\partial\,\tau} + O(\zeta^2) =\\
&= -\epsilon\,\Big(\bar{n}_{(1)(r)} + \frac{\partial\,\Psi_{(1)}^r}{\partial\,\tau}\Big)(\tau, \vec{\sigma}) + O(\zeta^2) =\\
&= -\epsilon\,\Big[\frac{\partial_r}{\triangle}\,{}^3K_{(1)} + \frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(4\,\mathcal{M}_r^{(UV)} - \frac{\partial_r}{\triangle}\sum_c \partial_c\,\mathcal{M}_c^{(UV)}\Big)\Big](\tau, \vec{\bar{\sigma}}) + O(\zeta^2),
\end{aligned}
$$

$$
\begin{aligned}
{}^4\bar{g}_{(1)rs}(\tau,\vec{\bar{\sigma}}) &= {}^4g_{rs}(\tau,\vec{\sigma}) - \epsilon\,\Big(\partial_r\,\Psi^s_{(1)} + \partial_s\,\Psi^r_{(1)}\Big)(\tau,\vec{\bar{\sigma}}) + O(\zeta^2) = \\
&= -\epsilon\,\Big(\delta_{rs}\,\Big[1 + 2\,(\Gamma^{(1)}_r + 2\,\phi_{(1)})\Big](\tau,\vec{\sigma}) + \Big[\partial_r\,\Psi^s_{(1)} + \partial_s\,\Psi^r_{(1)}\Big](\tau,\vec{\bar{\sigma}})\Big) + O(\zeta^2) = \\
&= {}^4\eta_{rs} + \Big[{}^4h^{TT}_{(1)rs} + \epsilon\,\frac{8\pi\,G}{c^3}\,\frac{\delta_{rs}}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}\Big](\tau,\vec{\sigma}) + O(\zeta^2).
\end{aligned}
\tag{6.14}
$$

In the last line we used Eq.(4.6) for $\phi_{(1)}$ to recover the TT 3-metric ${}^4h^{TT}_{(1)rs}$ of Eq.(6.7). As a consequence it turns out that the new 4-metric in the new radar 4-coordinates has the following expression

$$
{}^4\bar{g}_{AB} = {}^4\eta_{AB} + \epsilon\,\begin{pmatrix} -2\,\frac{\partial_\tau}{\triangle}\,{}^3K_{(1)} + \alpha(matter) & -\frac{\partial_r}{\triangle}\,{}^3K_{(1)} + \beta_r(matter) \\ -\frac{\partial_s}{\triangle}\,{}^3K_{(1)} + \beta_s(matter) & \epsilon\,{}^4h^{TT}_{(1)rs} + \delta_{rs}\,\gamma(matter) \end{pmatrix} + O(\zeta^2).
\tag{6.15}
$$

Therefore the coordinate transformation (6.13) leads to a *generalized TT-gauge* whose 3-metric is not 3-orthogonal due to the presence of the TT 3-metric. Also in absence of matter it differs from the usual harmonic ones, whose instantaneous 3-spaces are all Euclidean, for the non-spatial terms depending upon the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}(\tau,\vec{\sigma})$ (the HPM form of the gauge freedom in clock synchronization).

If the matter sources have a compact support and if the matter terms $\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}$ and $\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)r}$ are negligible in the radiation zone far away from the sources, then Eq.(6.15) gives a *spatial TT-gauge* with still the explicit dependence on the inertial gauge variable ${}^3\mathcal{K}_{(1)}$ (non existing in Newtonian gravity), which together with matter and the tidal variables, determines the non-Euclidean nature of the instantaneous 3-spaces.

D. Inversion of the Operator $\tilde{M}_{ab}$ and the Resulting Form of the Second Order Equations for $R_{\bar{a}}$.

Since Eq.(6.8) gives $\sum_b\,\tilde{M}_{ab}\,\Gamma^{(1)}_b = -\frac{\epsilon}{2}\,{}^4h^{TT}_{(1)aa}$ with the diagonal elements of the TT 3-metric satisfying the wave equation (6.11), i.e. $\Box\,{}^4h^{TT}_{(1)aa} \,{\buildrel \circ \over =}\, -\,\epsilon\,\frac{16\pi\,G}{c^3}\,T^{(TT)aa}_{(1)}$, to find the tidal variables $R_{\bar{a}} = \sum_a\,\gamma_{\bar{a}a}\,\Gamma^{(1)}_a$ associated to one solution of the wave equation we have to invert the operator $\tilde{M}_{ab}$.

If we introduce the functions

$$
\begin{aligned}
g_a &= -\frac{\epsilon}{2}\,{}^4h^{TT}_{(1)aa} = \sum_b\,\tilde{M}_{ab}\,\Gamma^{(1)}_b = \Gamma^{(1)}_a + \frac{1}{2}\,\Big(1 + \frac{\partial_a^2}{\Delta}\Big)\sum_c\,\frac{\partial_c^2}{\Delta}\,\Gamma^{(1)}_c - 2\frac{\partial_a^2}{\Delta}\,\Gamma^{(1)}_a = \\
&\sum_a\,g_a = 0, \qquad \Box\,g_a \,{\buildrel \circ \over =}\, \frac{8\pi\,G}{c^3}\,T^{(TT)aa}_{(1)},
\end{aligned}
\tag{6.16}
$$

we can write

$$\Gamma_a^{(1)} = g_a - \partial_a \Psi^a_{(1)} - \frac{1}{3} \sum_c \partial_c \Psi^c_{(1)}, \tag{6.17}$$

where $\Psi^a_{(1)}$ is the function generating the coordinate transformation (6.13).

If we define the quantities

$$\begin{aligned}
z_{ab}|_{a\neq b} &= {}^4h^{TT}_{(1)ab}|_{a\neq b} = -\frac{\partial_a\partial_b}{\Delta} \left(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c}{\Delta} \Gamma_c^{(1)} \right) = \\
&= 2 \left(\partial_a \Psi^b_{(1)} + \partial_b \Psi^a_{(1)} \right)|_{a\neq b}, \qquad (a \neq b),
\end{aligned}$$

$$\begin{aligned}
& \sum_b \partial_b\, {}^4h^{TT}_{(1)ab} = 0, \quad \Rightarrow \quad \sum_{b\neq a} \partial_b\, z_{ba} = -\partial_a\, g_a, \\
\Rightarrow \qquad & z_{ab}|_{a\neq b} = z_{ab}[g_c]|_{a\neq b}.
\end{aligned} \tag{6.18}$$

they turn out to be functionals of the g_a's expressible in terms of the functions $\Psi^a_{(1)}$ of Eq.(6.13). Therefore, also the functions $\Psi^a_{(1)}$ can be viewed as functionals $\Psi^a_{(1)}[g_r]$ of the g_a's.

As a consequence, the TT 3-metric of Eq.(6.7) can be written in the form

$$\begin{aligned}
{}^4h^{TT}_{(1)ab} &= -2\,\epsilon \left[\left(\Gamma_r^{(1)} + \sum_u \frac{\partial_u^2}{2\,\triangle} \Gamma_u^{(1)} \right) \delta_{rs} - \frac{\partial_r\,\partial_s}{\triangle} (\Gamma_r^{(1)} + \Gamma_s^{(1)}) + \frac{\partial_r\,\partial_s}{2\,\triangle} \sum_u \frac{\partial_u^2}{\triangle} \Gamma_u^{(1)} \right] = \\
&= \begin{pmatrix} g_1 & z_{12}[g] & z_{13}[g] \\ z_{12}[g] & g_2 & z_{23}[g] \\ z_{13}[g] & z_{23}[g] & g_3 \end{pmatrix}.
\end{aligned} \tag{6.19}$$

The transversality property of the TT metric, shown in Eq.(6.18), allows to express $z_{ab}[g_c]$ as the following functional of g_a

$$\begin{aligned}
z_{23}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2} \int^{\sigma^2,\sigma^3} d\tilde{\sigma}^2\, d\tilde{\sigma}^3\, \left[-\partial_3^2\, g_3 - \partial_2^2\, g_2 + \partial_1^2\, g_1 \right] (\tau, \sigma^1, \tilde{\sigma}^2, \tilde{\sigma}^3), \\
z_{13}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2} \int^{\sigma^1,\sigma^3} d\tilde{\sigma}^1\, d\tilde{\sigma}^3\, \left[-\partial_1^2\, g_1 - \partial_3^2\, g_3 + \partial_2^2\, g_2 \right] (\tau, \tilde{\sigma}^1, \sigma^2, \tilde{\sigma}^3), \\
z_{12}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2} \int^{\sigma^1,\sigma^2} d\tilde{\sigma}^1\, d\tilde{\sigma}^2\, \left[-\partial_2^2\, g_2 - \partial_1^2\, g_1 + \partial_3^2\, g_3 \right] (\tau, \tilde{\sigma}^1, \tilde{\sigma}^2, \sigma^3).
\end{aligned} \tag{6.20}$$

By using $\sum_a g_a = 0$, Eq(6.20) can be put in the following form

$$z_{ab}[g_c]|_{a\neq b}(\tau,\vec{\sigma}) = -\frac{1}{2}\int_{a\neq b}^{\sigma^a,\sigma^b} d\tilde{\sigma}^a\, d\tilde{\sigma}^b \left[(\Delta-\partial_a^2)\, g_b + (\Delta-\partial_b^2)\, g_a\right](\tau,\tilde{\sigma}^a,\tilde{\sigma}^b,\sigma^{c\neq a,b}). \quad (6.21)$$

Due to Eqs.(6.18), the functionals $\Psi^a_{(1)}[g_c]$ of the g_c's are related to the functionals given in Eqs.(6.21) by the equations

$$\Big(\partial_a\, \Psi^b_{(1)}[g_c] + \partial_b\, \Psi^a_{(1)}[g_c]\Big)|_{a\neq b} = z_{ab}[g_c]|_{a\neq b}. \quad (6.22)$$

The solution $\Psi^a_{(1)}[g_c]$ of this equation is (ϵ^2_{aef} is symmetric in e and f and vanishes if $a=e$ or $a=f$)

$$\begin{aligned}
\Psi^1_{(1)}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2}\int^{\sigma^2,\sigma^3} d\tilde{\sigma}^2\, d\tilde{\sigma}^3 \Big(\partial_3\, z_{12}[g_c] + \partial_2\, z_{13}[g_c] - \partial_1\, z_{23}[g_c]\Big)(\tau,\sigma^1,\tilde{\sigma}^2,\tilde{\sigma}^3),\\
\Psi^2_{(1)}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2}\int^{\sigma^1,\sigma^3} d\tilde{\sigma}^1\, d\tilde{\sigma}^3 \Big(\partial_3\, z_{23}[g_c] + \partial_2\, z_{12}[g_c] - \partial_1\, z_{12}[g_c]\Big)(\tau,\tilde{\sigma}^1,\sigma^2,\tilde{\sigma}^3),\\
\Psi^3_{(1)}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2}\int^{\sigma^1,\sigma^2} d\tilde{\sigma}^1\, d\tilde{\sigma}^2 \Big(\partial_1\, z_{23}[g_c] + \partial_3\, z_{12}[g_c] - \partial_2\, z_{13}[g_c]\Big)(\tau,\tilde{\sigma}^1,\tilde{\sigma}^2,\sigma^3),\\
&\quad or\\
\Psi^a_{(1)}[g_c](\tau,\vec{\sigma}) &= \frac{1}{2}\sum_{ef}(\epsilon_{aef})^2\int^{\sigma^e,\sigma^f} d\tilde{\sigma}^e\, d\tilde{\sigma}^f \Big(\partial_e\, z_{fa}[g_c] + \partial_f\, z_{ea}[g_c] - \partial_a\, z_{ef}[g_c]\Big)(\tau,\sigma^a,\tilde{\sigma}^e,\tilde{\sigma}^f).
\end{aligned} \quad (6.23)$$

By using Eq.(6.21) and $\sum_a g_a = 0$ we get the following form of Eqs.(6.23)

$$\begin{aligned}
\Psi^a_{(1)}[g_c](\tau,\vec{\sigma}) &= -\sum_{ef}\frac{(\epsilon_{aef})^2}{4}\int^{\sigma^e,\sigma^f} d\tilde{\sigma}^e d\tilde{\sigma}^f \int^{\tilde{\sigma}^e,\tilde{\sigma}^f} d\hat{\sigma}^e d\hat{\sigma}^f \int^{\sigma^a} d\tilde{\sigma}^a\\
&\Big[(\Delta-\partial_e^2)\Delta\, g_f + (\Delta-\partial_f^2)\Delta\, g_e - \partial_e^2\,\partial_f^2\,(g_e+g_f)\Big](\tau,\tilde{\sigma}^a,\hat{\sigma}^e,\hat{\sigma}^f).
\end{aligned} \quad (6.24)$$

Then by using Eqs.(6.17) we get

$$\begin{aligned}
\Gamma^{(1)}_a &= g_a - \partial_a\, \Psi^a_{(1)}[g_b] - \frac{1}{3}\sum_c \partial_c\, \Psi^c_{(1)}[g_b] =\\
&= \sum_{bc}\tilde{M}^{-1}_{ab}\,\tilde{M}_{bc}\,\Gamma^{(1)}_c = \sum_b \tilde{M}^{-1}_{ab}\, g_b,
\end{aligned} \quad (6.25)$$

and this is a definition of the inverse operator $\tilde{M}^{-1}_{ab}$ by means of Eq.(6.24).

If $g_c \stackrel{\circ}{=} \rho_c$ is a solution of the wave equation $\Box\, g_c \stackrel{\circ}{=} \frac{8\pi\, G}{c^3}\, T^{(TT)cc}_{(1)}$ (see Eqs.(6.11) and (6.16)), then the associated tidal variables (our physical radiative degrees of freedom replacing the TT 3-metric of the standard approach in harmonic gauges) are

$$R_{\bar{a}}\, =\, \sum_a \gamma_{\bar{a}a}\, \Gamma^{(1)}_a \stackrel{\circ}{=} \sum_{ab} \gamma_{\bar{a}a}\, \tilde{M}^{-1}_{ab}\, \rho_b = \sum_a \gamma_{\bar{a}a}\, \Big(\rho_a - \partial_a\, \Psi^a_{(1)}[\rho_b] - \frac{1}{3}\, \sum_c \partial_c\, \Psi^c_{(1)}[\rho_b]\Big), \quad (6.26)$$

with the functional $\Psi^a_{(1)}[\rho_b]$ of Eqs.(6.23). Eqs.(6.4), (6.10), (6.11) imply that these tidal variables are solutions of the wave equations

$$\begin{aligned}
\Box \sum_b \tilde{M}_{ab}\, \Gamma^{(1)}_b\, &\stackrel{\circ}{=}\, \frac{8\pi\, G}{c^3}\, T^{(TT)aa}_{(1)},\\
\Box \sum_{\bar{b}} M_{\bar{a}\bar{b}}\, R_{\bar{b}}\, &\stackrel{\circ}{=}\, \frac{8\pi\, G}{c^3}\, \sum_a \gamma_{\bar{a}a}\, T^{(TT)aa}_{(1)}.
\end{aligned} \qquad (6.27)$$

In the next Section we will study the solutions for the tidal variables, because they are the *HPM gravitational waves with asymptotic background* in our family of 3-orthogonal gauges with non-Euclidean 3-spaces Σ_τ.

From Eqs.(2.3), (3.6), and by using $\sum_{\bar{a}} \gamma_{\bar{a}a}\, \gamma_{\bar{a}b}\, =\, \delta_{ab} - \frac{1}{3}$, we get that the extrinsic curvature tensor of our 3-spaces in our family of 3-orthogonal gauges is the following first order quantity

$$ {}^3K_{(1)rs}\, =\, \sigma_{(1)(r)(s)}|_{r \not= s} + \delta_{rs}\, \Big(\frac{1}{3}\, {}^3K_{(1)} - \partial_\tau\, \Gamma^{(1)}_r + \partial_r\, \bar{n}_{(1)(r)} - \sum_a \partial_a\, \bar{n}_{(1)(a)}\Big), \qquad (6.28)$$

with $\bar{n}_{(1)(r)}$ and $\sigma_{(1)(r)(s)}|_{r \not= s}$ given in Eqs.(4.16) and (4.17), respectively (they depend on ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K$). Therefore, our (dynamically determined) 3-spaces have a first order deviation from Euclidean 3-spaces, embedded in the asymptotically flat space-time, determined by both instantaneous inertial matter effects and retarded tidal ones. Moreover the inertial gauge variable ${}^3K_{(1)}$ (non existing in Newtonian gravity) is still free. From Eqs.(2.5) we see that the intrinsic 3-curvature of these non-Euclidean 3-spaces is ${}^3\hat{R}|_{\theta^i=0} = 2\, \sum_a \partial_a^2\, \Gamma_a^{(1)}$: it is determined only by the tidal variables, i.e. by the HPM gravitational waves propagating inside these 3-spaces.

If we use the coordinate system of Eqs. (6.13) (6.15) to go in the generalized TT gauge, we can introduce the standard polarization pattern of gravitational waves for ${}^4h^{TT}_{rs}$ (see Refs.[20–22]) and then the inverse transformation allows to rewrite it in our family of 3-orthogonal gauges.

VII. POST-MINKOWSKIAN GRAVITATIONAL WAVES WITH ASYMPTOTIC BACKGROUND IN THE 3-ORTHOGONAL GAUGES

In this Section we study the solutions of the linearized equations for the tidal variables $R_{\bar{a}}(\tau,\vec{\sigma})$ and the TT 3-metric ${}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma})$, namely the PM gravitational waves with asymptotic background in the family of 3-orthogonal gauges. When needed we assume the validity of multipolar expansion of the energy-momentum tensor, which is reviewed in Appendix B.

A. The Retarded Solution

Since in Eqs.(6.27) we have the flat wave operator $\Box = \partial_\tau^2 - \triangle$ associated with the asymptotic Minkowski 4-metric, we give the solution as a retarded integral over the past flat null cone attached to the point $(\tau,\vec{\sigma})$ on the instantaneous 3-space Σ_τ at time τ by using the retarded Green function $G(\tau,\vec{\sigma};\tau^{'},\vec{\sigma}^{'}) = -\theta(\tau-\tau^{'})\,\frac{\delta(\tau-|\vec{\sigma}-\vec{\sigma}^{'}|-\tau^{'}])}{4\pi\,|\vec{\sigma}-\vec{\sigma}^{'}|}$ $(\Box\, G(\tau,\vec{\sigma};\tau^{'},\vec{\sigma}^{'}) = \delta(\tau^{'}-\tau)\,\delta^3(\vec{\sigma}-\vec{\sigma}^{'}))$.

The retarded solution of Eqs.(6.27) is

$$
\begin{aligned}
R_{\bar{a}}(\tau,\vec{\sigma}) \;&=\; \sum_a \gamma_{\bar{a}a}\,\Gamma_a^{(1)}(\tau,\vec{\sigma}) \stackrel{\circ}{=} \sum_{ab} \gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}(\tau,\vec{\sigma})\,\Big(F_b^{TT(hom)}(\tau,\vec{\sigma}) - \\
&-\; \frac{2\,G}{c^3}\int d^3\sigma_1\,\frac{T^{(TT)bb}_{(1)}(\tau-|\vec{\sigma}-\vec{\sigma}_1|;\vec{\sigma}_1)}{|\vec{\sigma}-\vec{\sigma}_1|}\Big),\\
&\Gamma_a^{(1)} = \sum_{\bar{a}} \gamma_{\bar{a}a}\,R_{\bar{a}},
\end{aligned}
\tag{7.1}
$$

where $F_a^{(hom)}(\tau,\vec{\sigma})$ is a homogeneous solution of the flat wave operator $(\Box\, F_a^{(hom)} = 0)$ and the non-local operator $\tilde{M}^{-1}_{ab}$ is defined by means of Eq.(6.26).

The condition of *no-incoming radiation* is $F_a^{(hom)}(\tau,\vec{\sigma}) = 0$: it uses the flat light-cone of the asymptotic Minkowski metric at $\tau \to -\infty$. Therefore there are only outgoing gravitational waves.

With our matter (point particles plus the electro-magnetic field) we do not need to solve the Hamilton equations independently outside and inside the matter sources with the subsequent matching of the solutions like it is usually done with compact matter sources. Even if $T^{(TT)aa}_{(1)}$ is assumed to have compact support, what is relevant in Eqs.(7.1) is the behavior far from the support of the non-local quantity determined by $\tilde{M}^{-1}_{ab}$ applied to the retarded integral.

Since we have $T^{TT}_{(1)\,ab}(\tau,\vec{\sigma}) = \mathcal{P}_{abcd}\,T^{cd}_{(1)}(\tau,\vec{\sigma})$, with the operator $\mathcal{P}_{abcd} = \mathcal{P}_{abcd}(\vec{\sigma})$ defined in Eq.(6.7), the chosen solution (7.1) has the following form

$$
\begin{aligned}
R_{\bar a}(\tau,\vec\sigma) &\stackrel{\circ}{=} -\frac{2G}{c^3}\sum_{ab}\gamma_{\bar a a}\,\tilde M^{-1}_{ab}(\tau,\vec\sigma)\int d\tau_1\,d^3\sigma_1\,\theta(\tau-\tau_1)\,\frac{\delta(\tau-|\vec\sigma-\vec\sigma_1|-\tau_1])}{|\vec\sigma-\vec\sigma_1|}\\
&\quad \sum_{uv}\mathcal{P}_{bbuv}(\vec\sigma_1)\,T^{uv}_{(1)}(\tau_1,\vec\sigma_1)=\\
&= -\frac{2G}{c^3}\sum_{ab}\gamma_{\bar a a}\,\tilde M^{-1}_{ab}(\tau,\vec\sigma)\int\frac{d^3\sigma_1}{|\vec\sigma-\vec\sigma_1|}\Big(\sum_{uv}\mathcal{P}_{bbuv}(\vec\sigma_1)\,T^{uv}_{(1)}(\tau,\vec\sigma_1)\Big)_{\tau\rightarrow\tau-|\vec\sigma-\vec\sigma_1|}
\end{aligned}
\tag{7.2}
$$

Analogously the solution for the TT 3-metric ${}^4h^{TT}_{(1)ab}$, satisfying the wave equation (6.11), is

$$
{}^4h^{TT}_{(1)rs}(\tau,\vec\sigma)\stackrel{\circ}{=}\epsilon\,-\frac{4\,G}{c^3}\int d^3\sigma_1\,\frac{\Big(\sum_{uv}\,\mathcal{P}_{rsuv}(\vec\sigma_1)\,T^{uv}_{(1)}(\tau,\vec\sigma_1)\Big)_{\tau\rightarrow\tau-|\vec\sigma-\vec\sigma_1|}}{|\vec\sigma-\vec\sigma_1|}. \tag{7.3}
$$

Let us remark that, since we have $\Box\,\mathcal{P}_{abcd}=\mathcal{P}_{abcd}\,\Box$, another solution of Eqs.(6.11) is (in general it is a solution differing from Eq.(7.3) by a homogeneous solution of the wave equation)

$$
{}^4\tilde h^{TT}_{(1)rs}(\tau,\vec\sigma) = -\epsilon\,\frac{4G}{c^3}\sum_{uv}\mathcal{P}_{rsuv}(\vec\sigma)\int d^3\sigma_1\,\frac{T^{uv}_{(1)}(\tau-\mid\vec\sigma-\vec\sigma_1\mid,\vec\sigma_1)}{\mid\vec\sigma-\vec\sigma_1\mid}. \tag{7.4}
$$

To compare Eq.(7.4) with Eq.(7.3) we need the following integral representation of the operator $\mathcal{P}_{rsuv}(\vec\sigma)$ [9]

$$
\begin{aligned}
f^{TT}_{rs}(\tau,\vec\sigma) &= \sum_{uv}\mathcal{P}_{rsuv}(\vec\sigma)\,f^{uv}(\tau,\vec\sigma)\stackrel{def}{=}\int d^3\sigma_1\sum_{uv}d^{TT}_{rsuv}(\vec\sigma-\vec\sigma_1)\cdot\,f^{uv}(\tau,\vec\sigma_1),\\
d^{TT}_{rsuv}(\vec\sigma-\vec\sigma_1) &= \frac{1}{2}\Big((\delta_{ru}\,\delta_{sv}+\delta_{rv}\,\delta_{su}-\delta_{rs}\,\delta_{uv})\,\delta^3(\vec\sigma-\vec\sigma_1)+\\
&+\sum_{ab}\Big[\delta_{ua}\,(\delta_{rv}\,\delta_{sb}+\delta_{sv}\,\delta_{rb})+\delta_{va}\,(\delta_{ru}\,\delta_{sb}+\delta_{su}\,\delta_{rb})-\delta_{ra}\,\delta_{sb}\,\delta_{uv}\Big]\\
&\quad\frac{\delta^{ab}\,|\vec\sigma-\vec\sigma_1|^2-3\,(\sigma^a-\sigma^a_1)\,(\sigma^b-\sigma^b_1)}{4\pi\,|\vec\sigma-\vec\sigma_1|^5}+\\
&+\ \delta_{rs}\int d^3\sigma_2\,\frac{\delta^{rs}\,|\vec\sigma-\vec\sigma_2|^2-3\,(\sigma^r-\sigma^r_2)\,(\sigma^s-\sigma^s_2)}{4\pi\,|\vec\sigma-\vec\sigma_2|^5}\\
&\quad\frac{\delta^{uv}\,|\vec\sigma_2-\vec\sigma_1|^2-3\,(\sigma^u_2-\sigma^u_1)\,(\sigma^v_2-\sigma^v_1)}{4\pi\,|\vec\sigma_2-\vec\sigma_1|^5}\Big),
\end{aligned}
\tag{7.5}
$$

[9] We have $\triangle\,\frac{1}{4\pi\,|\vec\sigma|}=-\delta^3(\vec\sigma)$, $\partial_r\,|\vec\sigma-\vec\sigma_1|=\frac{\sigma^r-\sigma^r_1}{|\vec\sigma-\vec\sigma_1|}$, $\partial_r\,\frac{1}{|\vec\sigma-\vec\sigma_1|}=\frac{\sigma^r-\sigma^r_1}{|\vec\sigma-\vec\sigma_1|^3}$, $\partial_r\,\partial_s\,\frac{1}{|\vec\sigma-\vec\sigma_1|}=\frac{\delta^{rs}\,|\vec\sigma-\vec\sigma_1|^2-3\,(\sigma^r-\sigma^r_1)\,(\sigma^s-\sigma^s_1)}{|\vec\sigma-\vec\sigma_1|^5}$, $\partial_r\,f(\tau-|\vec\sigma-\vec\sigma_1|)=-\frac{\sigma^r-\sigma^r_1}{|\vec\sigma-\vec\sigma_1|}\,\partial_\tau\,f(\tau-|\vec\sigma-\vec\sigma_1|)$.

where the integral kernel d^{TT}_{abcd} is a function only of the difference $\vec{\sigma} - \vec{\sigma}_1$.

As a consequence, the two retarded solutions (7.3) and (7.4) for the TT 3-metric with the no-incoming radiation condition take the form

$$
{}^4h^{TT}_{(1)\,rs}(\tau, \vec{\sigma}) = -\epsilon\, \frac{4G}{c^3} \int d^3\sigma_1 \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}_1 - \vec{\sigma}_2)\, \frac{T^{uv}_{(1)}(\tau - \mid \vec{\sigma} - \vec{\sigma}_1 \mid, \vec{\sigma}_2)}{\mid \vec{\sigma} - \vec{\sigma}_1 \mid}, \qquad (7.6)
$$

$$
{}^4\tilde{h}^{TT}_{(1)\,rs}(\tau, \vec{\sigma}) = -\epsilon\, \frac{4G}{c^3} \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma} - \vec{\sigma}_2) \int d^3\sigma_1\, \frac{T^{uv}_{(1)}(\tau - \mid \vec{\sigma}_2 - \vec{\sigma}_1 \mid, \vec{\sigma}_1)}{\mid \vec{\sigma}_2 - \vec{\sigma}_1 \mid}, \qquad (7.7)
$$

respectively. To make the comparison either an explicit form of the energy-momentum tensor or its multipolar expansion is needed.

B. PM Gravitational Waves with Asymptotic Background: Multipolar Expansion and the Quadrupole Emission Formula

To look for the PM relativistic quadrupole formula we need the multipolar expansion of the energy-momentum tensor in terms of relativistic Dixon multipoles expressed in our rest-frame instant form of dynamics. In Appendix B there is a review of such multipoles based upon Ref.[23]. See chapter 3 of Ref.[21] for a review of the standard multipole expansions used in the literature (see Refs. [24, 25]).

1. *The multipolar expansion for the tidal variables $R_{\bar{a}}$*

By using the multipolar expansion (B2) centered on the center of energy $w^\mu_E(\tau) = z^\mu(\tau, 0)$ ($\vec{\eta}(\tau) = 0$) and the operator $\mathcal{P}_{rsuv}$ defined in Eq.(6.7), from Eq.(7.2) we get by making integrations by parts (assumed valid)

$$
\begin{aligned}
&-\frac{2\,G}{c^3} \int d^3\sigma_1\, \frac{T^{(TT)bb}_{(1)}(\tau - |\vec{\sigma} - \vec{\sigma}_1|; \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} = \\
&= -\frac{2\pi\, G}{c^3} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{uvr_1..r_n} \int d^3\sigma_1\, \frac{q^{r_1..r_n|uv}(\tau - |\vec{\sigma} - \vec{\sigma}_1|)}{|\vec{\sigma} - \vec{\sigma}_1|} \\
&\Big(\delta_{au}\,\delta_{av} - \frac{1}{2}\,(1 - \frac{\partial^2_{1a}}{\triangle_1})\,\delta_{uv} + \frac{1}{2}\,(1 + \frac{\partial^2_{1a}}{\triangle_1})\,\frac{\partial_{1u}\,\partial_{1v}}{\triangle_1} - \\
&-(\delta_{au}\,\frac{\partial_{1v}}{\triangle_1} + \delta_{av}\,\frac{\partial_{1u}}{\triangle_1})\,\partial_{1a}\Big)\, \frac{\partial^n\, \delta^3(\vec{\sigma}_1)}{\partial\, \sigma_1^{r_1} ... \partial\, \sigma_1^{r_n}} =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2\,G}{c^3}\,\sum_{n=0}^{\infty}\,\frac{(-)^n}{n!}\,\sum_{r_1..r_n}\,\Big[\Big(\partial_{1r_1}...\partial_{1r_n}\,\frac{q^{r_1..r_n|aa}(\tau-|\vec{\sigma}-\vec{\sigma}_1|)}{|\vec{\sigma}-\vec{\sigma}_1|}\Big)|_{\vec{\sigma}_1=0} -\\
&-\;\frac{1}{2}\,\sum_u\,\Big(\partial_{1r_1}...\partial_{1r_n}\,\frac{q^{r_1..r_n|uu}(\tau-|\vec{\sigma}-\vec{\sigma}_1|)}{|\vec{\sigma}-\vec{\sigma}_1|}\Big)|_{\vec{\sigma}_1=0} +\\
&+\;\frac{1}{2}\,\sum_{uv}\,\Big(\partial_{1r_1}...\partial_{1r_n}\,\frac{q^{r_1..r_n|uv}(\tau-|\vec{\sigma}-\vec{\sigma}_1|)}{|\vec{\sigma}-\vec{\sigma}_1|}\Big)|_{\vec{\sigma}_1=0} -\\
&-\;\frac{1}{2}\,\sum_u\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\partial^2_{2a}\,\partial_{2r_1}...\partial_{2r_n}\,\frac{q^{r_1..r_n|uu}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)}{|\vec{\sigma}-\vec{\sigma}_2|} -\\
&-\;\frac{1}{2}\,\sum_{uv}\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\partial^2_{2a}\,\partial_{2r_1}...\partial_{2r_n}\,\frac{q^{r_1..r_n|uv}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)}{|\vec{\sigma}-\vec{\sigma}_2|} -\\
&-\;2\,\sum_u\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\int\frac{d^3\sigma_3}{4\pi\,|\vec{\sigma}_2-\vec{\sigma}_3|}\,\partial_{3a}\,\partial_{3u}\,\partial_{3r_1}...\partial_{3r_n}\,\frac{q^{r_1..r_n|au}(\tau-|\vec{\sigma}-\vec{\sigma}_3|)}{|\vec{\sigma}-\vec{\sigma}_3|}\Big].
\end{aligned}
\tag{7.8}
$$

The $n=0$ term has the following expression

$$
\begin{aligned}
&-\frac{2\,G}{c^3}\,\Big[\frac{q^{|aa}(\tau-|\vec{\sigma}|)-\frac{1}{2}\,\sum_u\,q^{|uu}(\tau-|\vec{\sigma}|)+\frac{1}{2}\,\sum_{uv}\,q^{|uv}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|} -\\
&-\;\frac{1}{2}\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\partial^2_{2a}\,\frac{\sum_u\,q^{|uu}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)+\sum_{uv}\,q^{|uv}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)}{|\vec{\sigma}-\vec{\sigma}_2|} -\\
&-\;2\,\sum_u\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\int\frac{d^3\sigma_3}{4\pi\,|\vec{\sigma}_2-\vec{\sigma}_3|}\,\partial_{3a}\,\partial_{3u}\,\frac{q^{|uu}(\tau-|\vec{\sigma}-\vec{\sigma}_3|)}{|\vec{\sigma}-\vec{\sigma}_3|} =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{G}{c^3}\,\Big[\frac{\partial^2_\tau\,q^{aa|\tau\tau}(\tau-|\vec{\sigma}|)-\frac{1}{2}\,\sum_u\,\partial^2_\tau\,q^{uu|\tau\tau}(\tau-|\vec{\sigma}|)+\frac{1}{2}\,\sum_{uv}\,\partial^2_\tau\,q^{uv|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|} -\\
&-\;\frac{1}{2}\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\partial^2_{2a}\,\frac{\sum_u\,\partial^2_\tau\,q^{uu|\tau\tau}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)+\sum_{uv}\,\partial^2_\tau\,q^{uv|\tau\tau}(\tau-|\vec{\sigma}-\vec{\sigma}_2|)}{|\vec{\sigma}-\vec{\sigma}_2|} -\\
&-\;2\,\sum_u\,\int\frac{d^3\sigma_2}{4\pi\,|\vec{\sigma}_2|}\,\int\frac{d^3\sigma_3}{4\pi\,|\vec{\sigma}_2-\vec{\sigma}_3|}\,\partial_{3a}\,\partial_{3u}\,\frac{\partial^2_\tau\,q^{uu|\tau\tau}(\tau-|\vec{\sigma}-\vec{\sigma}_3|)}{|\vec{\sigma}-\vec{\sigma}_3|} =\\
&= -\frac{G}{c^3}\,\frac{\sum_{uv}\,\mathcal{P}_{aauv}\,\partial^2_\tau\,q^{uv|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|} = -\frac{G}{c^3}\,\frac{\partial^2_\tau\,q^{(TT)aa|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|},\\
&\Rightarrow\qquad R_{\bar{a}}(\tau,\vec{\sigma}) = -\frac{G}{c^3}\,\sum_{ab}\,\gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}\,\frac{\partial^2_\tau\,q^{(TT)aa|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|} + (higher\,multipoles).
\end{aligned}
\tag{7.9}
$$

In the last expression we used Eq.(B13) and (B14) and $\sum_r\,\mathcal{P}_{rruv}=0$. The last line used Eq.(6.26).

The first term of Eq.(7.9) gives the standard *quadrupole emission* formula modified by the non-local operator $\tilde{M}^{-1}_{ab}$ of Eq.(6.26). The higher terms would give the contribution of the mass octupole, momentum quadrupole and so on (see Section 3.4 of Ref.[21]).

2. *The multipolar expansion for the TT 3-metric* ${}^4h^{TT}_{(1)rs}$

By putting in the solutions (7.6) and (7.7) the multipolar expansion (B2) with $\eta^r(\tau) = 0$ we get

$$\begin{aligned} {}^4h^{TT}_{(1)\,rs}(\tau, \vec{\sigma}) \;&=\; -\epsilon\, \frac{4G}{c^3} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int d^3\sigma_1 \sum_{r_1..r_n uv} \frac{q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma} - \vec{\sigma}_1 \mid)}{\mid \vec{\sigma} - \vec{\sigma}_1 \mid} \\ & \frac{\partial^n}{\partial \sigma_1^{r_1} ... \partial \sigma_1^{r_n}} d^{TT}_{rsuv}(\vec{\sigma}_1). \end{aligned} \tag{7.10}$$

$$\begin{aligned} {}^4\tilde{h}^{TT}_{(1)\,rs}(\tau, \vec{\sigma}) \;&=\; -\epsilon\, \frac{4G}{c^3} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma} - \vec{\sigma}_2) \sum_{r_1..r_n} \frac{\partial^n}{\partial \sigma_2^{r_1} ... \partial \sigma_2^{r_n}} \left(\frac{q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma}_2 \mid)}{\mid \vec{\sigma}_2 \mid} \right) = \\ &=\; {}^4h^{TT}_{(1)\,rs}(\tau, \vec{\sigma}). \end{aligned} \tag{7.11}$$

These two expressions can be shown to coincide with the change of variable $\vec{\sigma} - \vec{\sigma}_1 = \vec{\sigma}_2$ ($d^3\sigma_1 = d^3\sigma_2$) and by making suitable integrations by parts (we assume that the integrations by parts can be done).

We shall use Eq.(7.11), for whose manipulation we need the formula

$$\begin{aligned} & \frac{\partial^n}{\partial \sigma_2^{r_1} ... \partial \sigma_2^{r_n}} \left(\frac{q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma}_2 \mid)}{\mid \vec{\sigma}_2 \mid} \right) = \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\partial^k}{\partial \sigma_2^{r_1} ... \partial \sigma_2^{r_k}} \left(\frac{1}{\mid \vec{\sigma}_2 \mid} \right) \frac{\partial^{n-k} q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma}_2 \mid)}{\partial \sigma_2^{r_{k+1}} ... \partial \sigma_2^{r_n}} = \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-)^{n-k} \frac{\partial^k}{\partial \sigma_2^{r_1} ... \partial \sigma_2^{r_k}} \left(\frac{1}{\mid \vec{\sigma}_2 \mid} \right) \partial_\tau^{n-k} q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma}_2 \mid)\, n_{(2)\,r_{k+1}} ... n_{(2)\,r_n}, \\ & n_r = \frac{\sigma^r}{\mid \vec{\sigma} \mid}, \qquad n_{(2)\,r} = \frac{\sigma_2^r}{\mid \vec{\sigma}_2 \mid}. \end{aligned} \tag{7.12}$$

Therefore Eq.(7.11) can be put in the form

$$\begin{aligned} {}^4h^{TT}_{(1)\,rs}(\tau, \vec{\sigma}) \;&=\; -\epsilon\, \frac{4G}{c^3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-)^{2n-k}}{k!(n-k)!} \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma} - \vec{\sigma}_2) \times \\ & \sum_{r_1..r_n} \frac{\partial^k}{\partial \sigma_2^{r_1} ... \partial \sigma_2^{r_k}} \left(\frac{1}{\mid \vec{\sigma}_2 \mid} \right) \partial_\tau^{n-k} q^{r_1...r_n|uv}(\tau- \mid \vec{\sigma}_2 \mid)\, n_{(2)\,r_{k+1}} ... n_{(2)\,r_n}. \end{aligned} \tag{7.13}$$

To study the behavior of Eq.(7.13) at big distances, i.e. $r = |\vec{\sigma}| >> 1$, we use the following results shown in Appendix B of Ref.[26]

$$
\begin{aligned}
&f^{uv}(\vec{\sigma}) \mapsto \frac{A^{uv}}{r} \Rightarrow f^{TT}{}_{rs}(\vec{\sigma}) = \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}-\vec{\sigma}_2)\, f^{uv}(\vec{\sigma}_2) \mapsto \frac{B_{rs}}{r} + \mathcal{O}(1/r^3),\\
&f^{uv}(\vec{\sigma}) \mapsto \frac{A^{uv}}{r^2} \Rightarrow f^{TT}{}_{rs}(\vec{\sigma}) = \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}-\vec{\sigma}_2)\, f^{uv}(\vec{\sigma}_2) \mapsto \frac{B_{rs}}{r^2} + \mathcal{O}(1/r^3),\\
&f^{uv}(\vec{\sigma}) \mapsto \frac{A^{uv}}{r^n} \Rightarrow f^{TT}{}_{rs}(\vec{\sigma}) = \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}-\vec{\sigma}_2)\, f^{uv}(\vec{\sigma}_2) \mapsto \mathcal{O}(1/r^3), \qquad n \geq 3.
\end{aligned}
\tag{7.14}
$$

As a consequence at great distance only the terms with $k = 0$ give the dominant contribution in Eq.(7.13)

$$
\begin{aligned}
{}^4h^{TT}_{(1)\,rs}(\tau,\vec{\sigma}) \;=\; & -\epsilon\,\frac{4G}{c^3}\sum_{n=0}^{\infty}\int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}-\vec{\sigma}_2)\\
& \sum_{r_1..r_n} n_{(2)\,r_1}...n_{(2)\,r_n}\,\frac{\partial^n_\tau q^{r_1...r_n|uv}(\tau-\mid\vec{\sigma}_2\mid)}{\mid\vec{\sigma}_2\mid} + O(1/r^2).
\end{aligned}
\tag{7.15}
$$

As shown in Refs. [20] (see also Ref. [25]) we have the following result

$$
\begin{aligned}
f^{uv}(\vec{\sigma}) \;&=\; f^{uv}(\mid\vec{\sigma}\mid),\\
&\Downarrow\\
f^{TT}{}_{rs}(\vec{\sigma}) \;&=\; \int d^3\sigma_2 \sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}-\vec{\sigma}_2)\, f^{uv}(\mid\vec{\sigma}_2\mid) = \sum_{uv}\Lambda_{rsuv}(n)\, f^{uv}(\mid\vec{\sigma}\mid) + O(1/r^2),
\end{aligned}
\tag{7.16}
$$

where $\Lambda_{abcd}(n)$ is the algebraic projector (defined after Eq.(6.7) for plane wave solutions)

$$
\Lambda_{abcd}(n) = (\delta_{ac} - n_a\, n_c)\,(\delta_{bd} - n_b\, n_c) - \frac{1}{2}\,(\delta_{ab} - n_a\, n_b)\,(\delta_{cd} - n_c\, n_d), \qquad n_r = \frac{\sigma^r}{\mid\vec{\sigma}\mid}.
\tag{7.17}
$$

Therefore at great distances we can write

$$
{}^4h^{TT}_{(1)\,rs}(\tau,\vec{\sigma}) = -\epsilon\,\frac{4G}{c^3}\sum_{uv}\Lambda_{rsuv}(n)\sum_{n=0}^{\infty}\sum_{r_1..r_n} n_{r_1}...n_{r_n}\,\frac{\partial^n_\tau q^{r_1...r_n|uv}(\tau-\mid\vec{\sigma}\mid)}{\mid\vec{\sigma}\mid} + O(1/r^2).
\tag{7.18}
$$

Eq.(7.18) coincides with Eq.(3.34) of Ref.[21] [10] and the solution can be put in the form

$$
\begin{aligned}
{}^4h^{TT}_{(1)\,rs}(\tau,\vec{\sigma}) &= -\epsilon\,\frac{4G}{c^3}\sum_{uv}\Lambda_{rsuv}(n)\,\frac{q^{|uv}(\tau-\mid\vec{\sigma}\mid)}{\mid\vec{\sigma}\mid} + \\
&+\ (higher\, multipoles) + O(1/r^2).
\end{aligned}
\tag{7.19}
$$

Then Eq.(B13), i.e. $q^{|uv} = \frac{1}{2}\,\partial^2_\tau\, q^{uv|\tau\tau}$, leads to the standard quadrupole emission formula. Again one could evaluate the contribution of higher multipoles (mass octupole, momentum quadrupole,...) as in Ref.[21].

3. The Far Field of Time-Dependent Sources

Let us look at the far field expression of our linearized 4-metric, given in Eqs.(6.12) in 3-orthogonal gauges.

By assuming matter sources with compact support and by using the multipolar expansion of Appendix B and Eq.(7.9), this equation can be rewritten in the following form in the far wave zone ($r = |\vec{\sigma}|$)

$$
\begin{aligned}
{}^4g_{(1)\tau\tau} &= \epsilon\,\Big[1-\frac{\mathcal{A}}{r}-2\,\frac{\partial_\tau}{\triangle}\,{}^3K_{(1)}\Big], \qquad \mathcal{A} = \frac{8\pi\, G}{c^3}\,\Big(M_{(1)}c+\sum_i \eta_i\,\frac{\vec{\kappa}_i^2}{\sqrt{m_i^2c^2+\vec{\kappa}_i^2}}\Big),\\
{}^4g_{(1)\tau r} &= -\epsilon\,\Big[\mathcal{N}_r+\frac{\partial_r}{\triangle}\,{}^3K_{(1)}\Big],\\
&\qquad \mathcal{N}_r = -\frac{4\pi\, G}{c^3}\,\frac{(\vec{\sigma}\times\vec{j}_{(1)})^r}{r^3} + [\partial_\tau\, and\, \partial^2_\tau\,(mass\, quadrupole) + \\
&\qquad +(higher\, multipoles)],\\
{}^4g_{(1)rs} &= -\epsilon\,\delta_{rs}\,\Big[1+\frac{8\pi\, G}{c^3}\,\frac{M_{(1)}c}{r}+[\partial^2_\tau\,(mass\, quadrupole)+(higher\, multipoles)]\Big],
\end{aligned}
\tag{7.20}
$$

with the two asymptotic Poincare' charges $M_{(1)}c$ and $\vec{j}_{(1)}$ given in Eqs.(4.21) and Eq.(4.23) respectively. Eqs.(B7) and (B12) have been used to get the result for ${}^4g_{(1)\tau r}$. The last term in $\mathcal{A}$ has been evaluated by omitting the electro-magnetic field and is negligible in the non-relativistic limit. The shift function $\mathcal{N}_r$ gives the description of gravito-magnetism, Lense-Thirring effect included (see for instance chapter 6 of Ref.[2]), in the non-harmonic 3-orthogonal gauges.

We see that the results in the 3-orthogonal gauges are of the same type as in the standard harmonic gauges as can be seen by comparing Eq.(7.20) with Eqs. (13.30) and (13.32) of Ref. [27]. The only new terms are those involving the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$.

---

[10] In Ref.[21] it is shown that for small velocities inside the source of gravitational waves the temporal derivatives of the stress tensor multipoles are negligible giving terms of order $O(v^2/c^2)$.

Eqs.(7.20) are compatible with the (direction-independent) boundary conditions at spatial infinity for the 4-metric of our class of asymptotically flat space-times, given after Eqs. (2.7) of paper I (see also Eqs.(5.5) of Ref.[6]).

C. PM Gravitational Waves with Asymptotic Background: the Energy Balance

Having found the GW's of the HPM linearization of gravity, the next step is to check the energy balance: the energy emitted by matter in the form of GW's must be present in the gravitational field and there should be a back-reaction on matter.

In the standard approach with compact sources there are various way to evaluate the energy balance:

1) One can introduce the Landau-Lifschitz energy-momentum pseudo-tensor $t^{\mu\nu}_{LL}$ of the gravitational field and use the conservation law $\partial_\mu [-{}^4g\, (T^{\mu\nu} + t^{\mu\nu}_{LL})] \circeq 0$ to evaluate dE/dt in the far-field zone by using the quadrupolar approximation for GW's (see for instance Ref.[27]). This method is also used in the MPM + MPN approach of Refs.[24, 28], where the back-reaction (starting at the 2.5PN order) of the GW's on the source can be taken into account till 3.5PN.

2) The same results are obtained with ADM Hamiltonian methods in Ref. [29] by taking a time-average of the work done by the quadrupole radiation-reaction force appearing in the Hamilton equations of the particles. Here it is emphasized the analogy with electrodynamics due to the appearance of the analogue of the Schott term and it shown that there are no runaway solutions.

3) Instead in Ref.[21] the coarse-graining method is used to find an effective energy-momentum tensor for the gravitational field at the second order, from which the increase in the energy of the gravitational field due to the emission of GW is evaluated.

All these methods give the same result. The complications come from the problems of regularization of the gravitational self-force [16, 17] in the evaluation of the back reaction.

Here we will show that we can recover the standard result without having the gravitational self-force, due to the Grassmann regularization ($\eta_i^2\, m_i^2 = 0$) of the gravitational self-energies, by using the conservation of the ADM energy (4.21). The effect will result from interference terms $\eta_i\, m_i\, \eta_j\, m_j$ with $i \neq j$ like it happens for the Larmor formula of the electro-magnetic case if Grassmann-valued electric charges are used to regularize the electro-magnetic self-energies [30]. For this calculation we consider only point particles as matter ignoring the electro-magnetic field, because we want to make a comparison with the treatments with compact sources (the calculations with the electro-magnetic field should add the assumption that such a field is localized in compact regions).

By making some integrations by parts (assumed valid with our boundary conditions), the ADM energy (4.21) has the form ($\mathcal{M}^{(UV)}_{(2)}$ is given in Eq.(3.12))

$$
\begin{aligned}
\frac{1}{c}\,\hat{E}_{ADM} &= \int d^3\sigma\, \Big(\mathcal{M}^{(UV)}_{(1)} + \mathcal{M}^{(UV)}_{(2)} + \frac{2\pi\, G}{c^3}\, \mathcal{M}^{(UV)}_{(1)}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} - \\
&- \sum_c \mathcal{M}^{(UV)}_{(1)c}\, \Big[\frac{\partial_c}{\triangle}\, {}^3K_{(1)} - \frac{8\pi\, G}{c^3}\, \frac{1}{\triangle}\, \Big(\mathcal{M}^{(UV)}_{(1)c} - \frac{1}{4} \sum_d \frac{\partial_c\, \partial_d}{\triangle}\, \mathcal{M}^{(UV)}_{(1)d} \Big) \Big] + \\
&+ \frac{c^3}{16\pi G} \sum_{\bar{a}\bar{b}} \Big[\partial_\tau\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\, \partial_\tau\, R_{\bar{b}} + \sum_a \partial_a\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\, \partial_a\, R_{\bar{b}} \Big] \Big)(\tau, \vec{\sigma}) + O(\zeta^3).
\end{aligned}
\tag{7.21}
$$

Since we have:

1) $R_{\bar{a}} = \sum_a \gamma_{\bar{a}a}\, \Gamma^{(1)}_a$ $(\sum_r \Gamma^{(1)}_r = 0)$;

2) $\tilde{M}_{ab} = \sum_{\bar{a}\bar{b}} \gamma_{\bar{a}a}\, \gamma_{\bar{b}b}\, M_{\bar{a}\bar{b}}$ from Eq.(6.4);

3) $\sum_b \tilde{M}_{ab}\, \Gamma^{(1)}_b = -\frac{\epsilon}{2}\, {}^4h^{TT}_{(1)aa}$ from Eq.(6.8);

4) $\delta_{rs}\, \Gamma^{(1)}_r = -\frac{\epsilon}{2} \Big({}^4h_{(1)rs} - \frac{1}{3}\, \delta_{rs} \sum_v {}^4h_{(1)vv} \Big)$ from ${}^4h_{(1)rs} = -2\, \epsilon\, \delta_{rs}\, (\Gamma^{(1)}_r + 2\, \phi_{(1)})$ with $2\, \phi_{(1)} = -\frac{\epsilon}{6} \sum_v {}^4h_{(1)vv}$;

5) $\delta_{rs}\, \Gamma^{(1)}_r = -\frac{\epsilon}{2} \Big({}^4h^{TT}_{(1)rs} + \frac{1}{2}\, (\partial_r\, \epsilon_{(1)s} + \partial_s\, \epsilon_{(1)r}) + (\partial_r\, \partial_s - \frac{1}{3}\, \delta_{rs}\, \triangle)\, \lambda_{(1)} \Big)$ from Eq.(6.5);

6) $\sum_r {}^4h^{TT}_{(1)rr} = \sum_r \partial_r\, {}^4h^{TT}_{(1)rs} = 0$;

the term bilinear in the gradients of $R_{\bar{a}}$ in Eq.(7.21) can be manipulated in such a way that, after an integration by parts, the final form of Eq.(7.21) becomes

$$
\begin{aligned}
\frac{1}{c}\,\hat{E}_{ADM} &= \int d^3\sigma\, \Big(\mathcal{M}^{(UV)}_{(1)} + \mathcal{M}^{(UV)}_{(2)} + \frac{2\pi\, G}{c^3}\, \mathcal{M}^{(UV)}_{(1)}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} - \\
&- \sum_c \mathcal{M}^{(UV)}_{(1)c}\, \Big[\frac{\partial_c}{\triangle}\, {}^3K_{(1)} - \frac{8\pi\, G}{c^3}\, \frac{1}{\triangle}\, \Big(\mathcal{M}^{(UV)}_{(1)c} - \frac{1}{4} \sum_d \frac{\partial_c\, \partial_d}{\triangle}\, \mathcal{M}^{(UV)}_{(1)d} \Big) \Big] + \\
&+ \frac{c^3}{64\pi G} \sum_{rs} \Big[\Big(\partial_\tau\, {}^4h^{TT}_{(1)rs} \Big)^2 + \sum_c \Big(\partial_c\, {}^4h^{TT}_{(1)rs} \Big)^2 \Big] \Big)(\tau, \vec{\sigma}) + O(\zeta^3) = \\
&\stackrel{def}{=} \frac{1}{c} \int d^3\sigma\, \rho_{E(1+2)}(\tau, \vec{\sigma}) + O(\zeta^3),
\end{aligned}
\tag{7.22}
$$

where in the last line we introduced the (coordinate dependent) density $\rho_{E(1+2)}(\tau, \vec{\sigma})$ of the ADM energy $\hat{E}_{ADM}$ till the order $O(\zeta^2)$. This energy density is the sum of three terms: a matter term $\rho^{(matter)}_{E(1+2)}(\tau, \vec{\sigma})$, a radiation term $\rho^{(rad)}_{E(1+2)}(\tau, \vec{\sigma})$ (the GW) and an interaction term $\rho^{(int)}_{E(1+2)}(\tau, \vec{\sigma})$ (the interaction of GW's with matter: it controls both the emission and the back-reaction). Therefore we have

$$
\rho_{E(1+2)}(\tau, \vec{\sigma}) = \rho^{(matter)}_{E(1+2)}(\tau, \vec{\sigma}) + \rho^{(rad)}_{E(1+2)}(\tau, \vec{\sigma}) + \rho^{(int)}_{E(1+2)}(\tau, \vec{\sigma}),
$$

$$\rho^{(matter)}_{E(1+2)}(\tau,\vec{\sigma}) \; = \; \mathcal{M}_{(1)}(\tau,\vec{\sigma})\,c = \sum_i \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\,\eta_i\,c\,\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)},$$

$$\rho^{(rad)}_{E(1+2)}(\tau,\vec{\sigma}) \; = \; \frac{c^4}{64\pi\,G}\sum_{rs}\,\Big[\Big(\partal_\tau\,{}^4h^{TT}_{(1)rs}\Big)^2+\sum_c\Big(\partial_c\,{}^4h^{TT}_{(1)rs}\Big)^2\Big](\tau,\vec{\sigma}),$$

$$\begin{aligned}
\rho^{(int)}_{E(1+2)}(\tau,\vec{\sigma}) \; &= \; \mathcal{M}_{(2)}(\tau,\vec{\sigma})\,c+\Big(\frac{2\pi\,G}{c^2}\,\mathcal{M}^{(UV)}_{(1)}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)}-\\
&- \; \sum_c \mathcal{M}^{(UV)}_{(1)c}\,\Big[\frac{\partial_c}{\triangle}\,{}^3K_{(1)}-\frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(\mathcal{M}^{(UV)}_{(1)c}-\frac{1}{4}\sum_d\frac{\partial_c\,\partial_d}{\triangle}\,\mathcal{M}^{(UV)}_{(1)d}\Big)\Big]\Big)(\tau,\vec{\sigma}))=\\
&= \; \sum_i \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\,\eta_i\,\Big[c\sum_b\frac{\kappa^2_{ib}(\tau)}{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}}\,\Big(\Gamma^{(1)}_b+\frac{1}{2}\sum_d\frac{\partial_d^2}{\triangle}\,\Gamma^{(1)}_d\Big)(\tau,\vec{\eta}_i(\tau))-\\
&- \; c\sum_b\frac{\kappa^2_{ib}(\tau)}{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}}\,\frac{G}{4\pi}\sum_{j\neq i}\eta_j\int d^3\sigma\,\frac{\sqrt{m_j^2\,c^2+\vec{\kappa}_j^2(\tau)}}{|\vec{\eta}_i(\tau)-\vec{\sigma}|\,|\vec{\sigma}-\vec{\eta}_j(\tau)|}-\\
&- \; c\sum_b\kappa_{ib}(\tau)\,\Big(\frac{\partial_b}{\triangle}\,{}^3K_{(1)}\Big)(\tau,\vec{\eta}_i(\tau))+\frac{2\,G}{c^2}\sum_b\sum_{j\neq i}\eta_j\,\frac{\kappa_{ib}(\tau)\,\kappa_{jb}(\tau)}{|\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)|}+\\
&+ \; \frac{G}{8\pi\,c^2}\sum_b\kappa_{ib}(\tau)\sum_d\kappa_{id}(\tau)\sum_{j\neq i}\eta_j\int d^3\sigma\,\frac{1}{|\vec{\eta}_i(\tau)-\vec{\sigma}|}\,\partial_b\,\partial_d\,\frac{1}{|\vec{\sigma}-\vec{\eta}_j(\tau)|}-\\
&- \; \frac{G}{2\,c^2}\sum_{j\neq i}\eta_j\,\frac{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}\,\sqrt{m_j^2\,c^2+\vec{\kappa}_j^2(\tau)}}{\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)}\Big].
\end{aligned} \tag{7.23}$$

Let us divide the 3-space Σ_τ in two regions by means of a sphere S of big radius $R >> l_c$: a) an inner region $V_{(inner)}$ with a compact sub-region V_c of linear dimension l_c containing all the particles (and the electro-magnetic field if we would add it); b) an asymptotic far region $V_{(far)}$. Let $n^r = \sigma^r/|\vec{\sigma}|$ be a unit 3-vector.

Since we have $\hat{E}_{ADM} = \hat{E}^{V_{(far)}}_{ADM} + \hat{E}^{V_{(inner)}}_{ADM}$ and $\rho^{(matter)}_{E(1+2)}(\tau,\vec{\sigma})|_{\vec{\sigma}\in V_{far}} = \rho^{(int)}_{E(1+2)}(\tau,\vec{\sigma})|_{\vec{\sigma}\in V_{far}} = 0$, we get $\hat{E}^{V_{(far)}}_{ADM} = \int_{V_{(far)}} d^3\sigma\,\rho^{(rad)}_{E(1+2)}(\tau,\vec{\sigma})$.

Since $\hat{E}_{ADM}$ is a constant, we have $\frac{d\,\hat{E}_{ADM}}{d\tau} = 0$ so that we get

$$
\begin{aligned}
\frac{d\,\hat{E}_{ADM}^{V(inner)}}{d\tau} &= -\frac{d\,\hat{E}_{ADM}^{V(far)}}{d\tau} = -\int_{V_{(far)}} d^3\sigma\, \partial_\tau\, \rho^{(rad)}_{E(1+2)}(\tau,\vec{\sigma}) = \\
&= -\frac{c^4}{32\pi\, G} \int_{V_{(far)}} d^3\sigma \sum_{rs} \Big[\partial_\tau\, {}^4h^{TT}_{(1)rs}\, \partial^2_\tau\, {}^4h^{TT}_{(1)rs} + \sum_c \partial_c\, {}^4h^{TT}_{(1)rs}\, \partial_\tau\, \partial_c\, {}^4h^{TT}_{(1)rs}\Big](\tau,\vec{\sigma}) = \\
&= -\frac{c^4}{32\pi\, G} \int_{V_{(far)}} d^3\sigma \sum_{rs} \Big[\sum_c \partial_c \Big(\partial_\tau\, {}^4h^{TT}_{(1)rs}\, \partial_c\, {}^4h^{TT}_{(1)rs}\Big) + \partial_\tau\, {}^4h^{TT}_{(1)rs}\, \Box\, {}^4h^{TT}_{(1)rs}\Big](\tau,\vec{\sigma}) \circeq \\
&\circeq -\frac{c^4}{32\pi\, G} \int_{V_{(far)}} d^3\sigma \sum_{rs} \sum_c \partial_c \Big(\partial_\tau\, {}^4h^{TT}_{(1)rs}\, \partial_c\, {}^4h^{TT}_{(1)rs}\Big)(\tau,\vec{\sigma}) + O(1/r^2),
\end{aligned}
\tag{7.24}
$$

where we have done an integration by parts and used Eqs.(6.11) (their second member is zero because the non-local TT quantity $T^{(TT)rs}_{(1)}(\tau,\vec{\sigma})$ is of order $O(1/r^2)$ ($r = |\vec{\sigma}|$) in $V_{(far)}$).

If in the far region we use the solution (7.19), we have $\partial_c\, {}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) = -n^c\, \partial_\tau\, {}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) + O(1/r^2)$ and $\sum_c n^c\, \partial_c\, {}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) = -\partial_\tau\, {}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) + O(1/r^2)$ because $r = |\vec{\sigma}| > R >> l_c$. Therefore we get (the sphere S is parametrized with the angles θ and φ)

$$
\begin{aligned}
\frac{d\,\hat{E}_{ADM}^{V(inner)}}{d\tau} &= -\frac{d\,\hat{E}_{ADM}^{V(far)}}{d\tau} \circeq \\
&\circeq -\frac{c^4}{32\pi\, G} \int_{V_{(far)}} d^3\sigma \sum_c \partial_c \Big[n^c \sum_{rs} \Big(\partial_\tau\, {}^4h^{TT}_{(1)rs}\Big)^2\Big](\tau,\vec{\sigma}) + O(1/r^2) = \\
&= \frac{c^4}{32\pi\, G}\, R^2 \int_S d(\cos\theta)\, d\varphi \sum_{rs} \Big(\partial_\tau\, {}^4h^{TT}_{(1)rs}\Big)^2(\tau,\vec{\sigma}) + O(1/r^2) = \\
&\overset{(7.19)}{=} \frac{G}{2\pi\, c^2} \int_S d(\cos\theta)\, d\varphi \sum_{rsuv} \Lambda_{rsuv}(n)\, \partial_\tau\, q^{|rs}(\tau - R)\, \partial_\tau\, q^{|uv}(\tau - R) + O(1/r^2) = \\
&= \frac{G}{5\, c^2} \sum_{rs} [\partial^3_\tau\, q^{rs|\tau\tau}(\tau - R)]\, [\partial^3_\tau\, q^{uv|\tau\tau}(\tau - R)] + O(1/r^2).
\end{aligned}
\tag{7.25}
$$

where in the last line we used Eq.(B13) and $\int_S d(\cos\theta)\, d\varphi\, \Lambda_{rsuv}(n) = \frac{2\pi}{15}(11\, \delta_{ru}\, \delta_{sv} - 4\, \delta_{rs}\, \delta_{uv} + \delta_{rv}\, \delta_{su})$ (see Eq.(3.74) of Ref.[21]). The change of sign is due to the fact that the unit vector n^c is minus the normal to the sphere S.

Therefore Eq.(7.25) reproduces the standard result for the total radiated power also named the total gravitational luminosity $\mathcal{L}$ of the source (see for instance Eqs. (1.153) and (3.75) of Ref. [21]) [11].

---

[11] Since $\tau = c\, t$, Eq.(7.25) can be rewritten as $\frac{d\,\hat{E}_{ADM}^{V(inner)}}{dt} = \frac{G}{5\, c^7}\Big(\frac{\partial\, q^{rs|\tau\tau}}{\partial\, t^3}\Big)^2$. However, our mass quadrupole $q^{rs|\tau\tau}$ is equal to $c\, Q^{rs}$, where Q^{rs} is the mass quadrupole of Ref.[21]. As a consequence, we have $\frac{d\,\hat{E}_{ADM}^{V(inner)}}{dt} = \frac{G}{5\, c^5}\Big(\frac{\partial\, Q^{rs}}{\partial\, t^3}\Big)^2$ as in Ref.[21].

See Appendix C for the evaluation of the balance equations for the 3-momentum and the angular momentum by using the conservation of the corresponding ADM generators (4.22) and (4.23).

D. PM Gravitational Waves with Asymptotic Background: Detection with the Geodetic Deviation Equation

In Ref.[21] there is a full discussion of the detectors of gravitational waves and of the reference frames for the observers looking for them. After discussing the observers using local inertial frames (with Riemann normal coordinates) or freely falling frames (with Fermi normal coordinates), there is a discussion of observers using a TT frame (where particles at rest before the arrival of the gravitational wave remain at rest after their arrival) and of their use of the geodesic equation and of the geodetic deviation equation. Then there is a discussion of the proper detector frame used by experimentalists (in it locally one uses an Euclidean Newtonian 3-space): now the solution of the geodetic deviation equation, giving the coordinate displacement $\vec{\xi}(t)$ of a test mass of mass m induced by a gravitational wave (zero in the TT frame), can be put in the form $m\, \frac{d^2\, \xi^i(t)}{dt^2} = F^i$ with the Newton force $F^i = \frac{m}{2}\, \frac{d^2\, {}^4h^{TT}_{ij}}{dt^2}\, \xi^j$. However, it is possible to give a coordinate-independent description of the effect of a gravitational wave on a test mass by using a notion of proper distance between two nearby geodesics, see Refs.[20, 22].

All these approaches could be reproduced with our formalism. In this Subsection we give first a discussion of geodesics in our generalized TT gauge as our alternative to the TT frame and then we discuss the proper distance and the geodesic deviation equation.

1. *The Geodetic Equation for Test Particles*

Let us consider the behavior of a test particle in presence of a gravitational wave in the distant wave zone far away from the sources. To this end let ignore the sources by putting $\mathcal{M}^{(UV)}_{(1)} = 0$, $\mathcal{M}^{(UV)}_{(1)a} = 0$, $T^{ab}_{(1)} = 0$ (they are assumed to have compact support and to give vanishing terms $E_{\bar{a}} = 0$ in Eqs.(6.4) evaluated in the far zone) and consider a gravitational wave solution of the wave equation $\Box\, h^{TT}_{(1)aa}(\tau, \vec{\sigma}) \stackrel{\circ}{=} 0$, see Eq.(6.8).

In this case the 4-metric is given in Eq.(6.12) with all the matter terms omitted (so that we have $\phi_{(1)} = \frac{1}{4\,\Delta} \sum_c \frac{\partial_c^2}{\Delta}\, \Gamma_c^{(1)}$) and with ${}^3\mathcal{K}_{(1)} = 0$ (here we assume that this 3-orthogonal gauge is relevant near the detector).

By using the embedding $z^\mu(\tau, \vec{\sigma}) = \epsilon^\mu_A\, \sigma^A$ discussed in the Introduction, we describe the world-line $\tilde{x}^\mu(s) = \epsilon^\mu_A\, \sigma^A(s) = \epsilon^\mu_\tau\, \tau(s) + \epsilon^\mu_r\, \tilde{\eta}^r(s)$ of the test particle as a time-like geodesic with parameter s ($\frac{d^2\, \sigma^A(s)}{ds^2} + {}^4\Gamma^A_{BC}(\sigma^E(s))\, \frac{d\sigma^B(s)}{ds}\, \frac{d\sigma^C(s)}{ds} = 0$). If we choose for s the proper time of the test particle, we have $s = s(\tau) = \sqrt{\epsilon\, {}^4g_{(1)\tau\tau}(\tau(s), \vec{\tilde{\eta}}(s))}$ and $\tilde{x}^\mu(s(\tau)) = x^\mu(\tau) = \epsilon^\mu_A\, \eta^A(\tau) = \epsilon^\mu_\tau\, \tau + \epsilon^\mu_r\, \eta^r(\tau)$. We have

$$\frac{ds}{d\tau} = 1 + n_{(1)}(\tau, \eta^u(\tau)) + O(\zeta^2). \tag{7.26}$$

In the weak field approximation the geodesic equation becomes the following equation for $\vec{\eta}(\tau)$

$$
\begin{aligned}
\frac{d^2\eta^r(\tau)}{d\tau^2} &= -\left({}^4\Gamma^r_{(1)\tau\tau} + 2\,{}^4\Gamma^r_{(1)\tau u}\,\dot{\eta}^u + {}^4\Gamma^r_{(1)uv}\,\dot{\eta}^u\dot{\eta}^v\right) + \\
&+ \left({}^4\Gamma^\tau_{(1)\tau\tau} + 2\,{}^4\Gamma^\tau_{(1)\tau u}\,\dot{\eta}^u + {}^4\Gamma^\tau_{(1)uv}\,\dot{\eta}^u\dot{\eta}^v\right)\,\dot{\eta}^r + \frac{1}{L}O(\zeta^2). \qquad (7.27)
\end{aligned}
$$

Let us consider the following Christoffel symbol

$$
\begin{aligned}
\epsilon\,{}^4\Gamma^r_{(1)\tau\tau} &= \frac{\partial \bar{n}_{(1)}}{\partial \sigma^r} + \frac{\partial\, n_{(1)(r)}}{\partial \tau} + \frac{1}{L}O(\zeta^2) = \\
&= \partial^2_\tau\left[\frac{2}{\Delta}\,\partial_a\Gamma_a^{(1)} - \frac{\partial_a}{2\,\Delta}\sum_c \frac{\partial^2_c}{\Delta}\,\Gamma_c^{(1)}\right] + \frac{1}{L}O(\zeta^2) = \\
&= \partial^2_\tau\,\Psi^r_{(1)} + O(\zeta^2), \qquad (7.28)
\end{aligned}
$$

where we used the function $\Psi^r_{(1)}$ defined in Eq.(6.13).

As a consequence the geodetic equation can be written in the form

$$
\begin{aligned}
\frac{d^2}{d\tau^2}\left[\eta^r(\tau) - \Psi^r_{(1)}(\tau,\eta^r(\tau))\right] &= -2\left({}^4\Gamma^r_{(1)\tau u} - \frac{\partial \Psi^r_{(1)}}{\partial\tau\partial\sigma^u}\right)\dot{\eta}^u - \left({}^4\Gamma^r_{(1)uv} - \frac{\partial \Psi^r_{(1)}}{\partial\sigma^v\partial\sigma^u}\right)\dot{\eta}^u\dot{\eta}^v + \\
&+ \left({}^4\Gamma^\tau_{(1)\tau\tau} + 2\,{}^4\Gamma^\tau_{(1)\tau u}\,\dot{\eta}^u + {}^4\Gamma^\tau_{(1)uv}\,\dot{\eta}^u\dot{\eta}^v\right)\,\dot{\eta}^r + \frac{1}{L}O(\zeta^2). \quad (7.29)
\end{aligned}
$$

A special set of geodesics solution of Eq.(7.29) is *implicitly* defined by the condition

$$
\eta^r(\tau) - \Psi^r_{(1)}(\tau,\eta^r(\tau)) = \mathit{constant}. \qquad (7.30)
$$

In this case we have

$$
\dot{\eta}^r = \frac{\partial \Psi^r_{(1)}}{\partial\tau} + \frac{\partial \Psi^r_{(1)}}{\partial\sigma^s}\,\dot{\eta}^s, \qquad (7.31)
$$

so that in the weak field approximation we get

$$
\dot{\eta}^r = \frac{\partial \Psi^r_{(1)}}{\partial\tau} + O(\zeta^2) = O(\zeta). \qquad (7.32)
$$

As a consequence the right side of Eq.(7.29) is zero modulo terms of order $\frac{1}{L}\,O(\zeta^2)$: $\frac{d^2}{d\tau^2}\left[\eta^r(\tau) - \Psi^r_{(1)}(\tau,\eta^r(\tau))\right] = 0 + \frac{1}{L}\,O(\zeta^2)$.

Moreover in the weak field approximation the 4-velocity of the geodesic (7.30) turns out to be

$$
u^A = \frac{d\sigma^A(s)}{ds} = \frac{d\eta^A(\tau)}{d\tau}\,\frac{d\tau(s)}{ds} = \left(u^\tau = 1 - n_{(1)};\; u^r = \frac{\partial \Psi^r_{(1)}}{\partial\tau}\right). \qquad (7.33)
$$

Let us remark that, since we have $\dot{\eta}^r = \frac{\partial \Psi^r_{(1)}}{\partial \tau} + O(\zeta^2) = O(\zeta)$ along these geodesics, then we get $\frac{dA}{dt} = \frac{\partial A}{\partial \tau} + \frac{\partial A}{\partial \sigma^s}\, \dot{\eta}^s = \frac{\partial A}{\partial \tau} + \frac{1}{L}\, O(\zeta^2)$ for every quantity $A(\tau, \sigma^u)$ of order $O(\zeta)$.

The solution (7.30) selects a special family of geodesics whose meaning can be clarified by remembering the coordinate transformation (6.13) leading to a generalized TT gauge, whose associated 4-metric is given in Eq.(6.15) with the matter terms omitted.

The new Christoffel symbols ${}^4\overline{\Gamma}^A{}_{BC} = \frac{\partial \overline{\sigma}^A}{\partial \sigma^A} \left({}^4\Gamma^D{}_{EF} \frac{\partial \sigma^E}{\partial \overline{\sigma}^B} \frac{\partial \sigma^F}{\partial \overline{\sigma}^C} + \frac{\partial^2 \sigma^D}{\partial \overline{\sigma}^B \partial \overline{\sigma}^C} \right)$ imply

$$\overline{\Gamma}^r_{(1)\tau\tau} = 0 + \frac{1}{L}\, O(\zeta^2). \tag{7.34}$$

This equation is the sufficient and necessary condition to get the result that the coordinate lines $\overline{\sigma}^r = \mathit{constant}$ be geodetic. This consequence of the solution (7.26) is in accord with the choice of constant spatial coordinates for the usual TT gauge done in Ref. [20], pp. 13-16.

2. Detection of Gravitational Waves

As shown in Ref. [22] the main observable for the detection of gravitational waves is the *proper distance* between two nearby geodesics.

The geodetic deviation $\mathcal{E}^A(\tau)$ is the infinitesimal 4-vector orthogonal to the 4-velocity $u^A(\tau)$ of the reference geodesic: $u_A(\tau)\, \mathcal{E}^A(\tau) = 0$. If $\eta^A(\tau) = (\tau, \eta^r(\tau))$ is the reference geodesic and $\eta^A(\tau) + \mathcal{E}^A(\tau)$ the nearby one, the proper distance between them is the invariant

$$D(\tau) = \sqrt{{}^4g_{AB}(\tau, \eta^u(\tau))\, \mathcal{E}^A(\tau)\, \mathcal{E}^B(\tau)}. \tag{7.35}$$

Following Ref.[20] and consistently with the weak field approximation we assume $D(\tau) << L$, namely that the proper distance is less of the wavelength of the gravitational wave to be detected.

Given the geodetic deviation equation for $\mathcal{E}^A(\tau)$

$$(u^A(\tau)\, \nabla_A)\, (u^B(\tau)\, \nabla_B)\, \mathcal{E}^C(\tau) = -{}^4R^C{}_{EFG}(\eta^D(\tau))\, u^E(\tau)\, \mathcal{E}^F(\tau)\, u^G(\tau), \tag{7.36}$$

we get the following equation for the proper distance (a scalar quantity)

$$\begin{aligned}
(u^A(\tau)\, \nabla_A)^2\, D(\tau) &= \frac{d^2}{ds^2}\, \tilde{D}(s(\tau)) = \\
&= \Big(1 - 2\, \bar{n}_{(1)}(\eta^E(\tau)\Big)\, \frac{d^2 D(\tau)}{d\tau^2} - \frac{d\, \bar{n}_{(1)}(\eta^E(\tau))}{d\tau}\, \frac{d\, D(\tau)}{d\tau} + \frac{1}{L}\, O(\zeta^2) = \\
&= -{}^4R_{ABCD}(\eta^E(\tau))\, u^B(\tau)\, u^D(\tau)\, \frac{\mathcal{E}^A(\tau)\, \mathcal{E}^C(\tau)}{D(\tau)}.
\end{aligned} \tag{7.37}$$

In the weak field approximation Eq.(7.37) becomes

$$\left(1 - 2\,\bar{n}_{(1)}(\eta^E(\tau)\right)\frac{d^2}{d\tau^2}D(\tau) - \frac{d\,\bar{n}_{(1)}(\eta^E(\tau))}{d\tau}\,\frac{d}{d\tau}D(\tau) =$$

$$= -{}^4R_{(1)\tau r\tau s}(\tau, \eta^u(\tau))\,\frac{\mathcal{E}^r(\tau)\,\mathcal{E}^s(\tau)}{D(\tau)} + \frac{1}{L}\,O(\zeta^2), \tag{7.38}$$

with the following expression for the Riemann tensor implied by Eq.(6.12)

$$\begin{aligned} {}^4R_{(1)\tau s\tau r} &= -\frac{\partial^2\,n_{(1)}}{\partial\sigma^s\partial\sigma^r} - \frac{1}{2}\frac{\partial}{\partial\tau}\left(\frac{\partial\bar{n}_{(1)(r)}}{\partial\sigma^s} + \frac{\partial\bar{n}_{(1)(r)}}{\partial\sigma^s}\right) + \delta_{rs}\,\frac{\partial^2}{\partial\tau^2}\left(\Gamma_a^{(1)} + 2\phi_{(1)}\right) + \frac{1}{L^2}\,O(\zeta^2) = \\ &= -\frac{1}{2}\,\partial_\tau^2\,h^{TT}_{(1)rs} + \frac{1}{L^2}\,O(\zeta^2). \end{aligned} \tag{7.39}$$

By using the result $\frac{dA}{dt} = \frac{\partial A}{\partial\tau} + \frac{1}{L}\,O(\zeta^2)$, valid for every quantity $A(\tau, \sigma^u)$ of order $O(\zeta)$, we get the following final form of the equation for the proper distance

$$\left(1 - 2\,\bar{n}_{(1)}(\tau, \eta^u(\tau))\right)\frac{d^2}{d\tau^2}D(\tau) - \frac{d\,\bar{n}_{(1)}(\eta^E(\tau))}{d\tau}\,\frac{d}{d\tau}D(\tau) =$$

$$+\frac{1}{2}\,\frac{d^2}{d\tau^2}\,h^{TT}_{rs}(\tau, \eta^u(\tau))\,\frac{\mathcal{E}^r(\tau)\,\mathcal{E}^s(\tau)}{D(\tau)} + \frac{1}{L}\,O(\zeta^2). \tag{7.40}$$

To study Eq.(7.40) we follow the method of Ref.[20].

The structure of Eqs. (7.36), (7.37) or (7.40) suggests that the geodetic deviation $\mathcal{E}^A(\tau)$ may be parametrized as the sum of a constant deviation $\mathcal{E}^A_o$ plus a small corrective term determined by the weak gravitational field. To this end let us introduce an arbitrary constant direction with the unit constant 4-vector n^A, ${}^4g_{AB}\,n^A\,n^B = 1$, so that the constant part of the deviation is given by $\mathcal{E}^A_o = D_o\,n^A$ with $D_o = const.$. If $\delta\mathcal{E}^A(\tau)$, with $\frac{\delta\mathcal{E}^A(\tau)}{D_o} = O(\zeta)$, is the small corrective term, then the geodetic deviation is parametrized in the following form

$$\mathcal{E}^A(\tau) = D_o\left(n^A + \frac{\delta\mathcal{E}^A(\tau)}{D_o} + O(\zeta^2)\right). \tag{7.41}$$

The induced parametrization of the proper distance is

$$D(\tau) = D_o\left(1 + \frac{\delta D(\tau)}{D_o} + O(\zeta^2)\right), \tag{7.42}$$

with $\frac{\delta D(\tau)}{D_o} = {}^4g_{(1)AB}\,n^A\,\frac{\delta\mathcal{E}^B(\tau)}{D_o} = O(\zeta)$.

With these parametrizations Eq. (7.40) takes the form

$$\frac{d^2}{d\tau^2}\frac{\delta D(\tau)}{D_o} = +\frac{1}{2}\,\frac{d^2}{d\tau^2}\,h^{TT}_{(1)rs}(\tau, \eta^u(\tau))\,n^r\,n^s + \frac{1}{L^2}\,O(\zeta^2). \tag{7.43}$$

allowing to find the solution for $\frac{\delta D(\tau)}{D_o}$ at the lowest order.

As a consequence we get the following expression for the proper distance

$$D(\tau) = D_o + + \frac{1}{2}\, h^{TT}_{(1)rs}(\tau, \eta^u(\tau))\, n^r\, n^s + O(\zeta^2). \tag{7.44}$$

The choice of the direction n^r allows to discuss the effects of the polarization of the gravitational wave on the detectors as it is done in Ref.[20, 22].

VIII. CONCLUSIONS

We have defined a consistent HPM linearization of ADM tetrad gravity in the York canonical basis in the family of non-harmonic 3-orthogonal Schwinger time gauges parametrized by the numerical value ${}^3K_{(1)}(\tau, \vec{\sigma})$ of the York time, the inertial gauge variable describing the general relativistic remnant of the freedom in clock synchronization. The non-Euclidean instantaneous 3-spaces Σ_τ (a first order deformation of the Euclidean 3-spaces of inertial Minkowski frames) are dynamically determined except for for the value of the trace ${}^3K_{(1)}$ of their extrinsic curvature tensor. The 4-metric has an asymptotic Minkowski background at spatial infinity. PN expansions are avoided by introducing a ultraviolet cutoff.

The PM solutions, $\tilde{\phi}_{(1)}(\tau, \vec{\sigma}) = 1 + 6\,\phi_{(1)}(\tau, \vec{\sigma})$ (the 3-volume element), $1 + n_{(1)}(\tau, \vec{\sigma})$ (the lapse function), $\bar{n}_{(1)(r)}(\tau, \vec{\sigma})$ (the shift functions), $\sigma_{(1)(r)(s)}|_{r \neq s}(\tau, \vec{\sigma})$ (the non-diagonal elements of the shear of the Eulerian observers), of the constraints and of the Hamilton equations implying the preservation in τ of the gauge-fixings, have both an instantaneous action-at-a-distance part depending on the instantaneous value of the matter energy-momentum tensor and a part depending on the tidal variables. At the PM level all previous quantities do not depend on the York time ${}^3K_{(1)}(\tau, \vec{\sigma})$ but on the spatially non-local function ${}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma}) = \frac{1}{\triangle}\, {}^3K_{(1)}(\tau, \vec{\sigma})$ (it can be named non-local York time).

In these non-harmonic gauges two functions $\sum_{\bar{b}} M_{\bar{a}\bar{b}}(\vec{\sigma})\, R_{\bar{a}}(\tau, \vec{\sigma})$, with the operator $M_{\bar{a}\bar{b}}(\vec{\sigma})$ containing only spatial derivatives, of the tidal variables $R_{\bar{a}}(\tau, \vec{\sigma})$ satisfy a wave equation with the flat asymptotic d'Alambertian. It has been shown that the operator $M_{\bar{a}\bar{b}}(\vec{\sigma})$ contains the information for determining the TT part of the 3-metric on Σ_τ. By using a no-incoming radiation condition with respect to the flat asymptotic metric, we get a retarded solution for the tidal variables in terms of the matter energy-momentum tensor which describes PM TT GW's with asymptotic background propagating at the velocity of light c [12]. A multipolar expansion of the energy-momentum tensor in terms of Dixon multipoles allows to get the standard quadrupole emission formula as the dominant part. Also the rate of variation of the energy is correct due to the Grassmann regularization [13] of the self-energies of the point particles (no gravitational self-energy) due to the existence of the conserved ADM energy. Thereforall the main properties of GW's are reproduced in our HPM approach. Only the hereditary and memory tails (coming from the matching of MPM and MPN solutions as said in Appendix A) are missing at this order: to study them we have

[12] See Ref.[31] for the problem of the velocity of propagation of the gravitational field in general relativity and in bimetric theories.

[13] In the electro-magnetic case [15] the Grassmann-valued electric charges allow to find the effective potential (Coulomb plus Darwin) corresponding to the one-photon exchange Feynman diagram. Therefore the problem of electro-magnetic self-energies is pushed to the level of loop diagrams and becomes part of the problem of renormalization of QED. At the classical level the Grassmann regularization allows to make sense of the classical equations of motion (not well defined due to essential singularities at the charge location): the replacement of the Grassmann-valued electric charges with the standard electric charge is equivalent to a classical renormalization of scalar charged particles. With gravity the same mechanism is at work, except that we do not yet have an accepted renormalizible theory of quantum gravity. Our procedure is a classical regularization and in some sense we are identifying the effective potential connected to the one graviton exchange.

to go to higher orders in the HPM expansion (see Section IIIB), which should correspond to the MPM solutin in the far wave zone.

All these results lead to a PM solution (modulo the choice of the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$) for the gravitational field and identify a class of PM Einstein space-times.

These results are *gauge dependent* because the York time ${}^3K_{(1)}$ is an inertial gauge variable. Therefore we have to face the *gauge problem in general relativity*. The gauge freedom of space-time 4-diffeomorphisms implies that a gauge choice is equivalent to the choice of a set of 4-coordinates in the atlas of the space-time 4-manifold.

The standard approach to the gauge problem is to try to describe physical properties in terms of *gauge invariant* quantities, i.e. in terms of 4-scalars. At the Hamiltonian level, where the Hamiltonian gauge group (whose generators are the first-class constraints) is equivalent to the 4-diffeomorphism group of space-time only *on-shell* (i.e. on the solutions of Einstein equations; see for instance Refs.[5]), the Hamiltonian gauge-invariant (off-shell) quantities are the *Dirac observables* (DO), which have zero Poisson bracket with the constraints. If we would know the solution of the super-Hamiltonian and super-momentum constraints in the form $\hat{\phi} = \tilde{\phi} - F[\theta^i, \pi_{\tilde{\phi}}, R_{\bar{a}}, \Pi_{\bar{a}}, matter] \approx 0$, $\hat{\pi}_i^{(\theta)} = \pi_i^{(\theta)} - G_i[\theta^i, \pi_{\tilde{\phi}}, R_{\bar{a}}, \Pi_{\bar{a}}, matter] \approx 0$, we could look for a canonical transformation from the York canonical basis to a Shanmugadhasan basis containing n, $\bar{n}_{(r)}$, $\hat{\theta}^i$, $\hat{\pi}_i^{(\theta)} \approx 0$, $\hat{\phi} \approx 0$, $\hat{\pi}_{\hat{\phi}}$, $\hat{R}_{\bar{a}}$, $\hat{\Pi}_{\bar{a}}$, $DO_{(matter)}$. In this basis $\hat{\theta}^i$ and $\hat{\pi}_{\hat{\phi}}$, n, $\bar{n}_{(r)}$, would be the (primary and secondary) inertial gauge variables and $\hat{R}_{\bar{a}}$, $\hat{\Pi}_{\bar{a}}$, $DO_{(matter)}$ the DO's (non-local quantities as functions of the original variables). However no-one is able to find such a basis. Also the most recent works of Refs.[32] contain existence proofs but no workable algorithm for finding the Dirac observables of the gravitational field. Moreover it is not clear how many of these DO are 4-scalars. Hopefully the 3-scalar tidal DO's $\hat{R}_{\bar{a}}$, $\hat{\Pi}_{\bar{a}}$, may be replaced with two pairs of 4-scalar DO's connected with the eigenvalues of the Weyl tensor [33] by means of a canonical transformation (this conjecture is under investigation). For the transverse electro-magnetic field one expects that the final DO's can be chosen as tetrad-dependent 4-scalars by using the Newman-Penrose formalism [33].

On the other side at the experimental level the description of baryon matter is intrinsically coordinate-dependent, namely is connected with the conventions used by physicists, engineers and astronomers for the modeling of space-time.

A) The description of satellites around the Earth is done by means of NASA coordinates [34] either in ITRS (frame fixed on the Earth surface) or in GCRS (frame centered on the Earth center) (see Ref.[35]).

B) The description of planets and other objects in the Solar System uses BCRS (a quasi-inertial Minkowski frame, if perturbations from the Milky Way are ignored [14]), centered in the barycenter of the Solar System] and ephemerides (see Ref.[35]).

C) In astronomy the positions of stars and galaxies are determined from the data (luminosity, light spectrum, angles) on the sky as living in a 4-dimensional nearly-Galilei space-time with the celestial ICRS [36] frame considered as a "quasi-inertial frame" (all galactic

[14] Essentially it is defined as a *quasi-inertial system*, *non-rotating* with respect to some selected fixed stars, in Minkowski space-time with nearly-Euclidean Newton 3-spaces. The qualification *quasi-inertial* is introduced to take into account general relativity, where inertial frames exist only locally. It can also be considered as a PM space-time with 3-spaces having a very small extrinsic curvature.

dynamics is Newtonian gravity), in accord with the assumed validity of the cosmological and Copernican principles. Namely one assumes a homogeneous and isotropic cosmological Friedmann-Robertson - Walker solution of Einstein equations (the standard ΛCDM cosmological model). In it the constant intrinsic 3-curvature of instantaneous 3-spaces is nearly zero as implied by the CMB data[37], so that Euclidean 3-spaces (and Newtonian gravity) can be used. However, to reconcile all the data with this 4-dimensional reconstruction one must postulate the existence of dark matter and dark energy as the dominant components of the classical universe after the recombination 3-surface!

As a consequence of the dependence on coordinates of the description of matter, our proposal for solving the gauge problem in our Hamiltonian framework with non-Euclidean 3-spaces is to choose a gauge (i.e. a 4-coordinate system) in non-modified Einstein gravity which is in agreement with the observational conventions in astronomy. Since ICRS has diagonal 3-metric, our 3-orthogonal gauges are a good choice. We are left with the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$ (not existing in Newtonian gravity). The suggestion is to try to fix ${}^3\mathcal{K}_{(1)}$ in such a way to eliminate dark matter as much as possible, by reinterpreting it as a relativistic inertial effect induced by the shift from Euclidean 3-spaces to non-Euclidean ones (independently from cosmological assumptions). As a consequence, ICRS should be reformulated not as a *quasi-inertial* reference frame in Galilei space-time, but as a reference frame in a PM space-time with ${}^3K_{(1)}$ deduced from the data connected to dark matter: as a consequence *what is called dark matter would be an indicator of the non-Euclidean nature of 3-spaces as 3-submanifolds of space-time (extrinsic curvature effect), whose internal 3-curvature can be very small if it is induced by GW's.* Then automatically BCRS would be its quasi-Minkowskian approximation (quasi-inertial reference frame in Minkowski space-time) for the Solar System. This point of view could also be useful for the ESA GAIA mission (cartography of the Milky Way) [38] and for the possible anomalies inside the Solar System [39].

As a consequence in the third paper [18] we will study the PN expansion of our HPM solution for a system of point particles without electro-magnetic field. There we will determine the explicit dependence of the equations of motion of the particles, of the proper time of time-like observers, of the time-like and null geodesics, of the redshift of light and of the luminosity distance upon the time- and spatial-gradients of the non-local York time ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$) in the PM space-time. Then we will study the slow motion limit making a PN expansion at all $\frac{n}{2}PN$ orders with a detailed study of the order 0.5PN.

We will show that the main observational supports for the existence of dark matter (rotation curves of galaxies, mass of galaxies from the virial theorem and from gravitational lensing; see for instance Refs.[40]) can be translated in restrictions on the non-local York time ${}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma})$ at the location of galaxies. If we could find a global phenomenological parametrization ${}^3\mathcal{K}^{(phen)}_{(1)}(\tau, \vec{\sigma})$ over all the 3-universe (in our PM space-time and with some average for the τ-dependence), we could get a phenomenological determination of the York time ${}^3\mathcal{K}^{(phen)}_{(1)}(\tau, \vec{\sigma}) = \triangle\, {}^3\mathcal{K}^{(phen)}_{(1)}(\tau, \vec{\sigma})$ to be used as an observational clock synchronization to be used to define the 3-spaces of a PM ICRS.

Appendix A: The Standard Approach to Gravitational Waves by using Einstein's Equations in the Family of 4-Harmonic Gauges

The standard description of GW's (see for instance Ref.[21]) is done in the family of 4-harmonic gauges after a decomposition of the 4-metric ${}^4g_{\mu\nu}$ of asymptotically flat space-times in a flat Minkowski background ${}^4\eta_{\mu\nu}$ plus a small perturbation, ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}$, $|{}^4h_{\mu\nu}| << 1$. The linearized Einstein equations in harmonic gauges have the form $\Box\, {}^4\bar{h}_{\mu\nu} = -\epsilon\, \frac{16\pi\, G}{c^3}\, T_{\mu\nu}$, $\partial^\nu\, {}^4\bar{h}^{\mu\nu} = 0$ (${}^4\bar{h}_{\mu\nu} = {}^4h_{\mu\nu} - \frac{1}{2}\, {}^4\eta_{\mu\nu}\, {}^4h$, ${}^4h = {}^4\eta^{\mu\nu}\, {}^4h_{\mu\nu}$, $\epsilon = \pm$ according to the signature convention for the 4-metric), where $T_{\mu\nu}$ is the matter energy-momentum tensor [15] satisfying $\partial^\nu\, T_{\mu\nu} = 0$ at this order.

It turns out that with an allowed coordinate transformation inside the family of harmonic gauges, i.e. ${}^4\bar{h}_{\mu\nu} \mapsto {}^4\bar{h}^{'}_{\mu\nu} = {}^4\bar{h}_{\mu\nu} - (\partial_\mu\, \xi_\nu + \partial_\nu\, \xi_\mu - {}^4\eta_{\mu\nu}\, \partial_\rho\, \xi^\rho)$, one can identify a transverse-traceless (TT) harmonic gauge in which the only remnant of the gravitational field are the two polarizations of GW's.

This weak field description is assumed valid in the wave zone far away from the matter sources, which are assumed in slow motion (Post-Newtonian (PN) approximation $v << c$) and with a small self-gravity [16] (if d is the linear dimension of the source with mass M and $R_M = \frac{2\, G\, M}{c^2}$ its Schwarzschild radius, one has $\frac{R_M}{d} \approx (\frac{v}{c})^2 = \epsilon << 1$, i.e. $d >> R_M$; $\epsilon^{n/2} = (\frac{v}{c})^n$ is the $\frac{n}{2}\, PN$ order).

In this way one is led to the interpretation of a (spin 2) GW propagating in the Euclidean 3-spaces of an inertial frame in Minkowski space-time.

Since the equivalence principle forbids the existence of global inertial frames in Einstein space-times, the next step is to replace the above decomposition with one with respect to a curved background ${}^4\bar{g}_{\mu\nu}$, i.e. ${}^4g_{\mu\nu} = {}^4\bar{g}_{\mu\nu} + {}^4h^{(\bar{g})}_{\mu\nu}$, such that the GW's are only ripples on this background [17]. With this formalism and a coarse-grained description [18] one can evaluate the energy loss associated to the emission of GW's and take into account the back-reaction on the background which is therefore modified.

However the most advanced description of GW's is done with the Damour-Blanchet approach [24] or with the equivalent DIRE approach of Ref.[28] (see chapter 5 of Ref.[21] for an overall review; see also Ref.[20]). In these approaches one uses the decomposition ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}$ and writes a so-called relaxed form of Einstein equations for the quantity ${}^4\mathbf{h}^{\mu\nu} = \sqrt{|{}^4g|}\, {}^4g^{\mu\nu} - {}^4\eta^{\mu\nu}$, namely $\Box\, {}^4\mathbf{h}^{\mu\nu} = \epsilon\, \frac{16\pi\, G}{c^3}\, \tau^{\mu\nu}$, where the effective energy-momentum

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[15] Our $T_{\mu\nu}$ is $\frac{1}{c}\, T_{\mu\nu}$ of Ref.[21].

[16] See Ref.[21] for a review of the problem of self-gravity of extended compact objects, in particular binary systems, and of the effacement principle of the internal structure which becomes relevant only at the order 5PN.

[17] As shown in chapter 1 of Ref.[21] this decomposition makes sense if: A) either ${}^4\bar{g}_{\mu\nu}$ has a scale of spatial variation L_B and the wavelength of the GW is $\lambda << L_B$ (and $|{}^4h_{\mu\nu}| << \lambda/L_B << 1$); B) or ${}^4\bar{g}_{\mu\nu}$ contains only frequencies less than ν_B and the frequency of the GW is $\nu >> \nu_B$ (this case is the more relevant for detectors). Moreover, a detector of dimension L_D will react only to GW's with $\lambda >> L_D$. For GW's with frequency $10^{-4} - 1\, Hz$ (to be detected by LISA) the wave-length is of order $10^6 - 10^{10}\, cm$ ($\lambda\, \nu = c$). If the frequency is $1 - 10^4\, Hz$ (to be detected by LIGO, VIRGO), the wave-length is of order $10^{10} - 10^{14}\, cm$.

[18] Either a spatial average on many wavelengths λ of the GW or a temporal average on several periods $1/\nu$ of the GW (the method used in the detectors).

tensor is $\tau^{\mu\nu} = |{}^4g|\,T^{\mu\nu} + \frac{c^3}{16\pi\,G}\,\Lambda^{\mu\nu}({}^4\mathbf{h})$ with $\Lambda^{\mu\nu}({}^4\mathbf{h}) = \frac{16\pi\,G}{c^3}\,|{}^4g|\,t^{\mu\nu}_{LL} + \partial_\alpha\,{}^4\mathbf{h}^{\mu\beta}\,\partial_\beta\,{}^4\mathbf{h}^{\nu\alpha} - {}^4\mathbf{h}^{\alpha\beta}\,\partial_\alpha\,\partial_\beta\,{}^4\mathbf{h}^{\mu\nu}$. $t^{\mu\nu}_{LL}$ is the Landau-Lifschitz energy-momentum pseudo-tensor of the gravitational field, which satisfies the ordinary conservation law $\partial_\nu\,[\sqrt{|{}^4g|}\,(T^{\mu\nu} + t^{\mu\nu}_{LL})] = 0$ due to Einstein equations. The ordinary equations of motion for the matter (i.e. ${}^4\nabla_\nu\,T^{\mu\nu} = 0$) are obtained by restricting the solutions of the relaxed Einstein equations to the harmonic gauges by requiring $\partial_\nu\,{}^4\mathbf{h}^{\mu\nu} = 0$.

Since the matter is supposed to have compact support of size d and to be in slow motion ($\sqrt{\epsilon} = \frac{v}{c} \approx \sqrt{\frac{R_M}{d}} << 1$ and typically with a wavelength of GW's satisfying $\lambda >> d$), the solution of the relaxed Einstein equations is obtained in three steps:

A) In the far wave zone ($r >> \lambda$ and $d < r < \infty$), where $T^{\mu\nu} = 0$, one makes a Post-Minkowskian (PM) expansion ${}^4\mathbf{h}^{\mu\nu} = \sum_{n=1}^{\infty} G^n\,{}^4\mathbf{h}^{\mu\nu}_n$ and uses the restriction $\partial_\nu\,{}^4\mathbf{h}^{\mu\nu} = 0$ to harmonic gauges. The iterative PM solution (including a homogeneous solution of the wave equation) is expressed in terms of *retarded* (symmetric trace free or STF) *radiative multipoles* of the gravitational field. Only a finite number of multipoles in ${}^4\mathbf{h}^{\mu\nu}_1$ are taken into account to avoid the problem of the Green function of the wave operator, which would require the knowledge of ${}^4\mathbf{h}^{\mu\nu}_1$ also in the near region where the PM expansion does not hold. With this regularization a multipolar PM (MPM) solution is found with the property that by making a PN expansion one finds ${}^4\mathbf{h}^{00}_n = O(1/c^{2n})$, ${}^4\mathbf{h}^{oi}_n = O(1/c^{2n+1})$, ${}^4\mathbf{h}^{ij}_n = O(1/c^{2n})$.

B) In the near zone ($r << \lambda$; the exterior near zone is $d < r << \lambda$) one makes a PN expansion of ${}^4h_{\mu\nu}$ with ${}^4h_{00} = \sum_{n=1}^{\infty}\,{}^4h^{(2n)}_{00}$, ${}^4h_{0i} = \sum_{n=1}^{\infty}\,{}^4h^{(2n+1)}_{0i}$, ${}^4h_{ij} = \sum_{n=1}^{\infty}\,{}^4h^{(2n)}_{ij}$ with ${}^4h^{(n)}_{\mu\nu} = O((\frac{v}{c})^n)$ ($v/c \approx \sqrt{R_M/d}$) and of the energy-momentum tensor $T^{00} = \sum_{n=0}^{\infty}\,{}^{(2n)}T^{00}$, $T^{0i} = \sum_{n=1}^{\infty}\,{}^{(2n+1)}T^{0i}$, $T^{ij} = \sum_{n=1}^{\infty}\,{}^{(2n)}T^{ij}$. Since the Newtonian approximation corresponds to ${}^4h^{(2)}_{00}$, ${}^4h_{oi} = 0$, ${}^4h_{ij} = 0$, the 1PN order contains ${}^4h^{(4)}_{00}$, ${}^4h^{(3)}_{0i}$, ${}^4h^{(2)}_{ij}$, the 2PN order contains ${}^4h^{(2)}_{00}$, ${}^4h^{(3)}_{0i}$, ${}^4h^{(2)}_{ij}$, and so on. Beyond some order one finds divergences connected to the inversion of the Laplacian operator which require the introduction of a regularization of the Poisson integrals. This is due to the fact that it is not possible to rebuild a retarded solution from its expansion for small retardation without going outside the near zone (i.e. there are divergencies for $r \to \infty$). To avoid these problems one introduces a multipolar PN (MPN) expansion and uses it in the relaxed Einstein equations with ${}^4\mathbf{h}^{\mu\nu} = \sum_{n=2}^{\infty} \frac{1}{c^n}\,{}^4h^{\mu\nu}_{(n)}$, $\tau^{\mu\nu} = \sum_{n=-2}^{\infty} \frac{1}{c^n}\,\tau^{\mu\nu}_{(n)}$. By using a regularization prescription one finds retarded solutions (regular at $r = 0$) in terms of *STF matter multipoles*. Also a homogeneous solution of the wave equation is needed: its regularity at $r = 0$ requires that it is a half retarded minus half advanced solution. Finally to get the equations of motion of matter and their PN expansion till order 3.5PN one has to impose the harmonic gauge condition and to take into account the back-reaction from the emission of GW's: this introduces the lacking terms ${}^4h^{(2n+1)}_{00}$, ${}^4h^{(2n)}_{0i}$, ${}^4h^{(2n+1)}_{ij}$, and becomes relevant at the 2.5PN order $[O((\frac{v}{c})^5)]$.

C) Then one matches the MPM and the MPN solutions in the overlap of the near and far zones: this allows to express the radiative multipoles in terms of the matter ones. Now one can study the limit at future null infinity ($r \to \infty$ with $u = r - t/c$ fixed) to test the nature of GW's. At higher orders hereditary terms (tails starting from 1.5PN $[O((\frac{v}{c})^3)]$ and non-linear (Christodoulou) memory starting from 3PN (see Ref.[41] for a review) appear, showing that GW's propagate not only on the flat light-cone but also inside it (i.e. with all possible speeds $0 \leq v \leq c$): there is an instantaneous wavefront followed by a tail traveling at

lower speed (it arrives later and then fades away) and a persistent zero-frequency non-linear memory.

Today there is control on the solution and on the matter equations of motion till order 3.5PN $[O((\frac{v}{c})^7)]$ (for binaries see the review in chapter 4 of Ref.[21]) and well established connections with numerical relativity (see the review in Ref.[42]) especially for the binary black hole problem (see the review in Ref.[43]).

In Refs.[29, 44] there is a Hamiltonian approach starting from a PN expansion of the ADM formalism in suitable non-harmonic gauges (generalized isotropic ones with ${}^3K = 0$), which allow to recover the previous results in harmonic gauges till the order 3PN.

However in this formulation GW's propagate in the background Euclidean 3-space implied by the decomposition ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}$ and it is not clear how to visualize them in the non-Euclidean instantaneous 3-spaces of the global non-inertial frames implied by the equivalence principle.

Appendix B: The Multipole Moments of the Matter Energy-Momentum Tensor

In Ref.[23] there is a study of the relativistic Dixon multipoles [45] [19] of the energy-momentum tensor of relativistic matter systems (for instance point particles plus the electromagnetic field) in the rest-frame instant form of dynamics in Minkowski space-time. Since in HPM gravity in our asymptotically flat space-times and in its non-inertial rest-frame developed in this paper we have only small deviations from such a scheme, we can apply this formalism (without using the general relativistic Dixon multipoles [48]) to the energy-momentum tensor given in Eqs. (3.12), which gives rise to the ten internal Poincare' generators given in Eqs. (4.21) - (4.37) at lowest order $O(\zeta)$. We refer to chapter 3 of Ref.[21] for a review of the standard types of multipoles used for the study of gravitational waves from compact sources.

Moreover at the lowest order we have $\partial_A\, T^{AB}_{(1)} \circeq 0$, $\partial_A\, (T^{AB}_{(1)}\, \sigma^c - T^{AC}_{(1)}\, \sigma^B) \circeq 0$, with $T^{\tau\tau}_{(1)} = \mathcal{M}^{(UV)}_{(1)}$, $T^{\tau r}_{(1)} = \mathcal{M}^{(UV)}_{(1)r}$, see after Eq.(3.12).

Let $w^\mu(\tau) = z^\mu(\tau, \vec{\eta}(\tau))$ be the time-like world-line of an arbitrary *center of motion* of the particle system. On the instantaneous 3-space Σ_τ we can define the following Dixon multipoles of the matter energy-momentum tensor $T^{AB}_{(1)}(\tau, \vec{\sigma})$ with respect to the point $\vec{\eta}(\tau)$

$$q^{r_1...r_n\,|\,AB}(\tau) = \int d^3\sigma\, \Big(\sigma^{r_1} - \eta^{r_1}(\tau)\Big)...\Big(\sigma^{r_n} - \eta^{r_n}(\tau)\Big)\, T^{AB}_{(1)}(\tau, \vec{\sigma}), \tag{B1}$$

and the following multipolar expansion of $T^{AB}_{(1)}$

$$\begin{aligned} T^{AB}_{(1)}(\tau, \vec{\sigma}) &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{r_1...r_n} q^{r_1...r_n\,|\,AB}(\tau)\, \frac{\partial^n\, \delta^3(\vec{\sigma}, \vec{\eta}(\tau))}{\partial\, \sigma^{r_1}...\partial\, \sigma^{r_n}},\\ T^{(TT)ab}_{(1)}(\tau, \vec{\sigma}) &= \sum_{uv} \Big(\mathcal{P}_{abuv}\, T^{uv}_{(1)}\Big)(\tau, \vec{\sigma}), \end{aligned} \tag{B2}$$

where Eq.(6.9) has to be used for the TT tensor.

To connect them to the standard Dixon multipoles of the space-time energy-momentum tensor $T^{\mu\nu}(x = z(\tau, \vec{\sigma})) = \Big[z^\mu_A\, z^\nu_B\, T^{AB}_{(1)}\Big](\tau, \vec{\sigma}) + O(\zeta^2)$ we must use the adapted embedding $z^\mu(\tau, \vec{\sigma})$ discussed in the Introduction.

By using Eqs.(4.21)-(4.37), the relevant multipoles are:

1a) the *mass monopole*

$$q^{|\,\tau\tau} = \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) = M_{(1)}\, c; \tag{B3}$$

1b) the *mass dipole* (Eq.(4.37) is used)

[19] As shown in this paper, strictly speaking the multipolar expansion makes sense only if the energy-momentum tensor is an analytic function of the 3-coordinates. However see Ref.[46] for the relaxation of this condition.

$$
\begin{aligned}
q^{r\,|\,\tau\tau} &= \int d^3\sigma\, (\sigma^r - \eta^r(\tau))\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) = j^{\tau r}_{(1)} - M_{(1)}\, c\, \eta^r(\tau) \approx \\
&\approx -M_{(1)}\, c\, \eta^r(\tau);
\end{aligned}
\tag{B4}
$$

1c) the *mass quadrupole* (Eq.(4.37) is used)

$$
\begin{aligned}
q^{rs\,|\,\tau\tau} &= \int d^3\sigma\, (\sigma^r - \eta^r(\tau))\, (\sigma^s - \eta^s(\tau))\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) \approx \\
&\approx \int d^3\sigma\, \sigma^r\, \sigma^s\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) + M_{(1)}\, c\, \eta^r(\tau)\, \eta^s(\tau);
\end{aligned}
\tag{B5}
$$

2a) the *momentum monopole* (Eq.(4.22) is used)

$$
q^{|\,\tau r} = \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)r}(\tau, \vec{\sigma}) = p^r_{(1)} \approx 0;
\tag{B6}
$$

2b) the *momentum dipole*

$$
q^{r\,|\,\tau s} = \int d^3\sigma\, (\sigma^r - \eta^r(\tau))\, \mathcal{M}^{(UV)}_{(1)s}(\tau, \vec{\sigma}) \approx \int d^3\sigma\, \sigma^r\, \mathcal{M}^{(UV)}_{(1)s}(\tau, \vec{\sigma}),
\tag{B7}
$$

whose antisymmetric part is $q^{r\,|\,\tau s} - q^{s\,|\,\tau r} = -2\, j^{rs}_{(1)}$ due to Eq.(4.23) [as a consequence we have $q^{r\,|\,\tau s} = \frac{1}{2}\,(q^{r\,|\,\tau s} + q^{s\,|\,\tau r}) - j^{rs}_{(1)}$];

3) the *stress tensor monopole*

$$
q^{|\,rs} = \int d^3\sigma\, T^{rs}_{(1)}(\tau, \vec{\sigma}) = q^{|\,sr}.
\tag{B8}
$$

If we choose as center of motion of the mass distribution the *center of energy* $w^\mu_E(\tau)$, we must put equal to zero the mass dipole and this implies $\eta^r(\tau) \approx 0$ (namely $w^\mu_E(\tau) = z^\mu(\tau, 0)$ coincides with the origin of 3-coordinates)

$$
\begin{aligned}
& q^{r\,|\,\tau\tau} \approx 0, \qquad \Rightarrow \qquad \eta^r(\tau) \approx 0, \\
& \Rightarrow \quad q^{rs|\tau\tau} = \int d^3\sigma\, \sigma^r\, \sigma^s\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}).
\end{aligned}
\tag{B9}
$$

In this case the non-zero lowest multipoles are the mass monopole $M_{(1)}\, c$, the mass quadrupole $q^{rs\,|\,\tau\tau}$, the momentum dipole $q^{r\,|\,\tau s}$ and the stress tensor monopole $q^{|\,rs}$.

The lowest order conservation law $\partial_A\, T^{AB}_{(1)} \circeq 0$ gives $\partial_\tau\, \mathcal{M}^{(UV)}_{(1)} \circeq -\sum_r \partial_r\, \mathcal{M}^{(UV)}_{(1)r}$ and $\partial_\tau\, \mathcal{M}^{(UV)}_{(1)r} \circeq -\sum_s \partial_s\, T^{rs}_{(1)}$, see Eqs.(3.13). By integrating over the whole 3-space with the matter density having compact support (or suitable fall-off at spatial infinity) we get

$$\partial_\tau\, M_{(1)} \stackrel{\circ}{=} 0 + O(\zeta^2), \qquad \partial_\tau\, p^r_{(1)} \stackrel{\circ}{=} 0 + O(\zeta^2). \tag{B10}$$

The conservation law $\partial_A\, (T^{AB}_{(1)}\, \sigma^c - T^{AC}_{(1)}\, \sigma^B) \stackrel{\circ}{=} 0$ implies

$$\partial_\tau\, j^{rs}_{(1)} \stackrel{\circ}{=} 0 + O(\zeta^2), \qquad \partial_\tau\, j^{\tau r}_{(1)} \stackrel{\circ}{=} 0 + O(\zeta^2). \tag{B11}$$

By using $\partial_A\, [T^{AB}_{(1)}\, \sigma^r\, \sigma^s] \stackrel{\circ}{=} T^{rB}_{(1)}\, \sigma^s + T^{sB}_{(1)}\, \sigma^r$ we get $\partial_\tau\, q^{rs\,|\,\tau\tau} \stackrel{\circ}{=} q^{r\,|\,\tau s} + q^{s\,|\,\tau r}$, so that we have

$$q^{r\,|\,\tau s} = \frac{1}{2}\, \partial_\tau\, q^{rs\,|\,\tau\tau} - j^{rs}_{(1)}. \tag{B12}$$

By using $\partial_A\, \partial_B\, (T^{AB}_{(1)}\, \sigma^C\, \sigma^D) \stackrel{\circ}{=} 2\, T^{CD}_{(1)}$ we get

$$2\, q^{|\,rs} \stackrel{\circ}{=} \partial^2_\tau\, q^{rs\,|\,\tau\tau}. \tag{B13}$$

Therefore the relevant multipoles are expressible in terms of $M_{(1)}$, $j^{rs}_{(1)}$, $\partial_\tau\, q^{rs\,|\,\tau\tau}$ and $\partial^2_\tau\, q^{rs\,|\,\tau\tau}$, where $q^{rs\,|\,\tau\tau}$ is the mass quadrupole.

However in Eq.(6.11) we have $T^{(TT)rs}_{(1)}$, whose stress tensor monopole is $q^{(TT)|\,rs} = \int d^3\sigma\, T^{(TT)rs}_{(1)}(\tau, \vec{\sigma})$, and not $T^{rs}_{(1)}$. From Eq.(6.9) we have $T^{(TT)rs}_{(1)} = \sum_{uv} \mathcal{P}_{rsuv}\, T^{uv}_{(1)}$. As a consequence we get $q^{(TT)|\,rs} = \int d^3\sigma\, T^{(TT)rs}_{(1)}(\tau, \vec{\sigma}) = \int d^3\sigma\, \sum_{uv} \mathcal{P}_{rsuv}\, T^{uv}_{(1)}(\tau, \vec{\sigma})$.

By using Eq.(6.9) we get (the surface terms vanish with the assumed support and boundary conditions)

$$\begin{aligned}
q^{|\,rs} &= \int d^3\sigma\, T^{rs}_{(1)}(\tau, \vec{\sigma}) = \int d^3\sigma\, \Big[\frac{1}{3}\, \tilde{H}_{(1)}\, \delta^{rs} + T^{(TT)rs}_{(1)}\Big](\tau, \vec{\sigma}) = \\
&= q^{(TT)|\,rs} + \frac{1}{3}\, \delta^{rs}\, \sum_u q^{|\,uu}, \\
\Rightarrow \qquad q^{(TT)|\,rs} &= q^{|\,rs} - \frac{1}{3}\, \delta^{rs}\, \sum_u q^{|\,uu} \stackrel{\circ}{=} \frac{1}{2}\, \partial^2_\tau\, (q^{rs\,|\,\tau\tau} - \frac{1}{3}\, \delta^{rs}\, \sum_u q^{uu\,|\,\tau\tau}).
\end{aligned} \tag{B14}$$

Therefore the monopole of the TT stress tensor $T^{(TT)rs}_{(1)}$ is connected to the second time derivative of the mass quadrupole as usual.

We also need higher multipoles $q^{r_1..r_n|rs} \approx \int d^3\sigma\, \sigma^{r_1} ... \sigma^{r_n}\, T^{rs}_{(1)}(\tau, \vec{\sigma})$ and their TT analogues. For the dipole we get

$$\int d^3\sigma\, \partial_A\, \partial_B\, \Big(\sigma^{r_1}\, \sigma^r\, \sigma^s\, T^{AB}_{(1)}(\tau, \vec{\sigma})\Big) = \partial^2_\tau\, q^{r_1 rs|\tau\tau} = 2\, \Big(q^{r_1|rs} + q^{r|r_1 s} + q^{s|r_1 r}\Big). \tag{B15}$$

We see that the second time derivative of the mass octupole is connected to a combination of stress tensor dipoles. For $r = s = a$ we have $\partial_\tau^2\, q^{r_1 aa|\tau\tau} = 2\,(q^{r_1|aa} + 2\,q^{a|r_1 a})$. However $q^{r_1|aa}$ cannot be expressed only in terms of the mass octupole.

The same pattern holds for the higher multipoles $q^{r_1..r_n|rs}$. In particular with same methods used in Ref.[21] we get the following results for the stress tensor monopole $q^{|rs}$ and dipole $q^{u|rs}$

$$
\begin{aligned}
& q^{|rs} = q^{|sr} = \partial_\tau\, q^{r|\tau s}, \\
& q^{r|us} + q^{s|ru} = \partial_\tau\, q^{rs|\tau u}, \\
& \partial_\tau\, q^{r|us} + \partial_\tau\, q^{s|ur} = \partial_\tau^2\, q^{rs|\tau u}, \\
& 2\,(\partial_\tau^{u|rs} + \partial_\tau\, q^{r|su} + \partial_\tau\, q^{s|ur}) = \partial_\tau^3\, q^{rsu|\tau\tau}, \\
& \partial_\tau\, q^{u|rs} = \frac{1}{6}\,\partial_\tau^3\, q^{rsu|\tau\tau} + \frac{1}{3}\,(\partial_\tau^2\, q^{ur|\tau s} + \partial_\tau^2\, q^{us|\tau r} - 2\,\partial_\tau^2\, q^{rs|\tau u}).
\end{aligned}
\tag{B16}
$$

in terms of the momentum dipole $q^{r|\tau u}$ and quadrupole $q^{rs|\tau u}$ and mass octupole $q^{rsu|\tau\tau}$.

Appendix C: The Balance of Momentum and Angular Momentum for GW's

Here we give the evaluation of the balance equations for the 3-momentum and the angular momentum by using the conservation of the corresponding ADM generators (4.22) and (4.23).

1. The Balance of Momentum for the GW's

By using Eqs.(6.4) and (6.8) the first term in $p^r_{(2)ADM}$ in Eq.(4.22) can be written in the form

$$\int d^3\sigma \quad \Big(\sum_{\bar a\bar b} \partial_r\, R_{\bar a}\, M_{\bar a\bar b}\, \partial_\tau\, R_{\bar b}\Big)(\tau,\vec\sigma) = \int d^3\sigma\, \Big(\sum_{ab} \partial_r\, \Gamma^{(1)}_a\, \tilde M_{ab}\, \Gamma^{(1)}_b\Big)(\tau,\vec\sigma) =$$
$$= \int d^3\sigma\, \Big(\sum_{rs} \partial_r\, {}^4h^{TT}_{(1)rs}\, \partial_\tau\, {}^4h^{TT}_{(1)rs}\Big)(\tau,\vec\sigma). \tag{C1}$$

Then, like in Eq.(7.23) for the ADM energy, the ADM momentum (4.22) can be written in the form

$$\begin{aligned}
\hat P^r_{(1+2)ADM} &= \int d^3\sigma\, \Big(\rho^{(matter)\,r}_{p(1+2)} + \rho^{(rad)\,r}_{p(1+2)} + \rho^{(int)\,r}_{p(1+2)}\Big)(\tau,\vec\sigma),\\
\rho^{(matter)\,r}_{p(1+2)}(\tau,\vec\sigma) &= \mathcal{M}^{(UV)}_{(1)r}(\tau,\vec\sigma) = \sum_i \delta^3(\vec\sigma - \vec\eta_i(\tau))\, \eta_i\, \kappa_{ir}(\tau),\\
\rho^{(rad)\,r}_{p(1+2)}(\tau,\vec\sigma) &= \frac{c^3}{32\pi\, G} \sum_{uv} \Big(\partial_r\, {}^4h^{TT}_{(1)uv}\, \partial_\tau\, {}^4h^{TT}_{(1)uv}\Big)(\tau,\vec\sigma),\\
\rho^{(int)\,r}_{p(1+2)}(\tau,\vec\sigma) &= \Big(\mathcal{M}^{(UV)}_{(1)r}\, \Big(\sum_a \frac{\partial^2_a}{\triangle}\, \Gamma^{(1)}_a - 2\, \Gamma^{(1)}_r\Big) + \frac{1}{2} \sum_{as} \mathcal{M}^{(UV)}_{(1)s}\, \partial_r\, \partial_s\, \frac{\partial^2_a}{\triangle}\, \Gamma^{(1)}_a -\\
&- \frac{8\pi\, G}{c^3}\, \mathcal{M}^{(UV)}_{(1)r}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} - \mathcal{M}^{(UV)}_{(1)}\, \frac{\partial_r}{\triangle}\, {}^3K_{(1)}\Big)(\tau,\vec\sigma).
\end{aligned} \tag{C2}$$

Like after Eq.(7.23) let us divide the 3-space Σ_τ in two regions by means of a sphere S of big radius $R >> l_c$: a) an inner region $V_{(inner)}$ with a compact sub-region V_c of linear dimension l_c containing all the particles (and the electro-magnetic field if we would add it); b) an asymptotic far region $V_{(far)}$. Let $n^r = \sigma^r/|\vec\sigma|$ be a unit 3-vector.

Since we have $\hat P^r_{ADM} = \hat P^{r\,V_{(far)}}_{ADM} + \hat P^{r\,V_{(inner)}}_{ADM}$ and $\rho^{(matter)\,r}_{p(1+2)}(\tau,\vec\sigma)|_{\vec\sigma\in V_{far}} = \rho^{(int)\,r}_{p(1+2)}(\tau,\vec\sigma)|_{\vec\sigma\in V_{far}} = 0$, we get $\hat P^{r\,V_{(far)}}_{ADM} = \int_{V_{(far)}} d^3\sigma\, \rho^{(rad)\,r}_{p(1+2)}(\tau,\vec\sigma)$.

Since $\hat P^r_{ADM} \approx 0$ is a constant, we have $\frac{d\,\hat P^r_{ADM}}{d\tau} = 0$ so that we get the following result in place of Eq.(7.24)

$$\begin{aligned}\frac{d\,\hat{P}^{r\,V_{(inner)}}_{ADM}}{d\tau} &= -\frac{d\,\hat{P}^{r\,V_{(far)}}_{ADM}}{d\tau} = -\int_{V_{(far)}} d^3\sigma\,\partial_\tau\,\rho^{r\,(rad)}_{p(1+2)}(\tau,\vec{\sigma}) = \\ &= -\frac{c^3}{32\pi\,G}\int_{V_{(far)}} d^3\sigma\sum_{rs}\Big[\partial_r\,{}^4h^{TT}_{(1)rs}\,\partial^2_\tau\,{}^4h^{TT}_{(1)rs} + \partial_r\,\partial_\tau\,{}^4h^{TT}_{(1)rs}\,\partial_\tau\,{}^4h^{TT}_{(1)rs}\Big](\tau,\vec{\sigma}).\end{aligned} \tag{C3}$$

By using the implication $\partial_\tau\,{}^4h^{TT}_{(1)rs} = -n^c\,\partial_c\,{}^4h^{TT}_{(1)rs} + O(1/R^2)$ ($n^c = \sigma^c/|\vec{\sigma}|$) of Eqs.(7.19), Eqs (7.25) are replaced by the following expression

$$\begin{aligned}\frac{d\,\hat{P}^{r\,V_{(inner)}}_{ADM}}{d\tau} &= -\frac{d\,\hat{P}^{r\,V_{(far)}}_{ADM}}{d\tau} = \\ &= -\frac{c^3}{32\pi\,G}\int_{V_{(far)}} d^3\sigma\sum_{rs}\sum_{c}\partial_c\Big(n^c\,\partial_\tau\,{}^4h^{TT}_{(1)rs}\,\partial_r\,{}^4h^{TT}_{(1)rs}\Big)(\tau,\vec{\sigma}) + O(1/R^2) = \\ &= \frac{c^3}{32\pi\,G}\,R^2\int_S d(cos\,\theta)\,d\varphi\sum_{rs}\Big(\partial_\tau\,{}^4h^{TT}_{(1)rs}\,\partial_r\,{}^4h^{TT}_{(1)rs}\Big)(\tau,\vec{\sigma}) + O(1/R^2) = \\ &\overset{(7.19)}{=} \frac{G}{2\pi\,c^3}\int_S d(cos\,\theta)\,d\varphi\sum_{swuv}\Lambda_{swuv}(n)\,\partial_\tau\,q^{|sw}(\tau-R)\,\partial_r\,q^{|uv}(\tau-R) + O(1/r^2) = \\ &= \frac{G}{8\,c^2}\int_S d(cos\,\theta)\,d\varphi\sum_{swuv}\Lambda_{swuv}(n)\,[\partial^3_\tau\,q^{sw|\tau\tau}(\tau-R)]\,[\partial_r\,\partial^2_\tau\,q^{uv|\tau\tau}(\tau-R)] + O(1/r^2),\end{aligned} \tag{C4}$$

after having used $q^{|rs} = \frac{1}{2}\,\partial^2_\tau\,q^{rs|\tau\tau}$, see Eq.(B13). This is the standard result for the momentum balance of GW's (see for instance Eqs. (1.164) and (3.83) of Ref.[21]).

2. The Balance of Angular Momentum for GW's

By using Eqs.(6.4) and (6.8) the last two lines of $j^{rs}_{(2)ADM}$ in Eq.(4.23) can be written in the form (see Eq.(2.51) of Ref.[21], where the angular momentum is defined with an overall minus signa with respect to us)

$$\begin{aligned}\frac{c^3}{64\pi\,G} &\int d^3\sigma\,\Big(\sum_{uv}\partial_\tau\,{}^4h^{TT}_{(1)uv}\,(\sigma^r\,\partial_s - \sigma^s\,\partial_r)\,{}^4h^{TT}_{(1)uv} - \\ &- \sum_u ({}^4h^{TT}_{(1)ru}\,\partial_\tau\,{}^4h^{TT}_{(1)su} - {}^4h^{TT}_{(1)su}\,\partial_\tau\,{}^4h^{TT}_{(1)ru})\Big)(\tau,\vec{\sigma}).\end{aligned} \tag{C5}$$

Then, like in Eq.(7.23) for the ADM energy, the ADM angular momentum (4.23) can be written in the form

$$
\begin{aligned}
\hat{J}^{rs}_{(1+2)ADM} &= \int d^3\sigma\, \Big(\rho^{(matter)\, rs}_{j(1+2)} + \rho^{(rad)\, rs}_{j(1+2)} + \rho^{(int)\, rs}_{j(1+2)}\Big)(\tau, \vec{\sigma}),\\
\rho^{(matter)\, rs}_{j(1+2)}(\tau, \vec{\sigma}) &= \sigma^r\, \mathcal{M}^{(UV)}_{(1)s}(\tau, \vec{\sigma}) - \sigma^s\, \mathcal{M}^{(UV)}_{(1)r}(\tau, \vec{\sigma}),\\
\rho^{(rad)\, rs}_{j(1+2)}(\tau, \vec{\sigma}) &= \frac{c^3}{64\pi\, G}\Big[\sum_{uv}\Big(\partial_\tau\, {}^4h^{TT}_{(1)uv}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, {}^4h^{TT}_{(1)uv}\Big) -\\
&- \sum_u \Big({}^4h^{TT}_{(1)ru}\, \partial_\tau\, {}^4h^{TT}_{(1)su} - {}^4h^{TT}_{(1)su}\, \partial_\tau\, {}^4h^{TT}_{(1)ru}\Big)\Big](\tau, \vec{\sigma}),\\
\rho^{(int)\, rs}_{j(1+2)}(\tau, \vec{\sigma}) &= \Big(\frac{1}{2}\, \mathcal{M}^{(UV)}_{(1)}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{1}{\triangle}\, {}^3K_{(1)} -\\
&- 4\Big[\sigma^r\, \mathcal{M}^{(UV)}_{(1)s}\, (\Gamma^{(1)}_r - \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} + \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c) -\\
&- \sigma^s\, \mathcal{M}^{(UV)}_{(1)r}\, (\Gamma^{(1)}_r - \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)} + \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c)\Big] +\\
&+ \sum_u \mathcal{M}^{(UV)}_{(1)u}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{\partial_u}{\triangle}\, \Gamma^{(1)}_u - \frac{1}{4}\sum_c (\mathcal{M}^{(UV)}_{(1)r}\, \frac{\partial_s}{\triangle} - \mathcal{M}^{(UV)}_{(1)s}\, \frac{\partial_r}{\triangle})\, \frac{\partial_c^2}{\triangle}\, \Gamma^{(1)}_c -\\
&- \frac{1}{4}\sum_{uv} \mathcal{M}^{(UV)}_{(1)u}\, (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, \frac{\partial_u}{\triangle}\, \frac{\partial_v^2}{\triangle}\, \Gamma^{(1)}_v + \mathcal{M}^{(UV)}_{(1)r}\, \frac{\partial_r}{\triangle}\, \Gamma^{(1)}_s - \mathcal{M}^{(UV)}_{(1)s}\, \frac{\partial_s}{\triangle}\, \Gamma^{(1)}_r\Big)(\tau, \vec{\sigma}).
\end{aligned}
\tag{C6}
$$

Like after Eq.(7.23) let us divide the 3-space Σ_τ in two regions by means of a sphere S of big radius $R >> l_c$: a) an inner region $V_{(inner)}$ with a compact sub-region V_c of linear dimension l_c containing all the particles (and the electro-magnetic field if we would add it); b) an asymptotic far region $V_{(far)}$. Let $n^r = \sigma^r / |\vec{\sigma}|$ be a unit 3-vector.

Since we have $\hat{J}^{rs}_{ADM} = \hat{J}^{rs\, V_{(far)}}_{ADM} + \hat{J}^{rs\, V_{(inner)}}_{ADM}$ and $\rho^{(matter)\, rs}_{j(1+2)}(\tau, \vec{\sigma})|_{\vec{\sigma} \in V_{far}} = \rho^{(int)\, rs}_{j(1+2)}(\tau, \vec{\sigma})|_{\vec{\sigma} \in V_{far}} = 0$, we get $\hat{J}^{rs\, V_{(far)}}_{ADM} = \int_{V_{(far)}} d^3\sigma\, \rho^{(rad)\, rs}_{j(1+2)}(\tau, \vec{\sigma})$.

Since $\hat{J}^{rs}_{ADM}$ is a constant, we have $\frac{d\, \hat{J}^{rs}_{ADM}}{d\tau} = 0$ so that we get the following result in place of Eq.(7.24) (again by using Eqs.(7.19) and $n^c = \sigma^c / |\vec{\sigma}|$)

$$
\begin{aligned}
\frac{d\, \hat{J}^{rs\, V_{(inner)}}_{ADM}}{d\tau} &= -\frac{d\, \hat{J}^{rs\, V_{(far)}}_{ADM}}{d\tau} = -\int_{V_{(far)}} d^3\sigma\, \partial_\tau\, \rho^{rs\, (rad)}_{j(1+2)}(\tau, \vec{\sigma}) =\\
&= \int_{V_{(far)}} d^3\sigma\, \Big[n^c\, \partial_c\, \rho^{rs\, (rad)}_{j(1+2)}\Big](\tau, \vec{\sigma}) + O(1/R^2) =\\
&= \int_{V_{(far)}} d^3\sigma\, \partial_c\, \Big[n^c\, \rho^{rs\, (rad)}_{j(1+2)}\Big](\tau, \vec{\sigma}) + O(1/R^2).
\end{aligned}
\tag{C7}
$$

As a consequence, we get

$$\begin{aligned}
\frac{d\,\hat{J}^{rs\,V_{(inner)}}_{ADM}}{d\tau} &= -\frac{d\,\hat{J}^{rs\,V_{(far)}}_{ADM}}{d\tau} = \\
&= \frac{c^3}{12\pi\,G}\,R^2\,\int_S d(cos\,\theta)\,d\varphi\Big[\sum_{uv}\,\Big(\partial_\tau\,{}^4h^{TT}_{(1)uv}\,(\sigma^r\,\partial_s - \sigma^s\,\partial_r)\,{}^4h^{TT}_{(1)uv}\Big) - \\
&- \sum_u\,\Big({}^4h^{TT}_{(1)ru}\,\partial_\tau\,{}^4h^{TT}_{(1)su} - {}^4h^{TT}_{(1)su}\,\partial_\tau\,{}^4h^{TT}_{(1)ru}\Big)\Big](\tau,\vec{\sigma}) + O(1/R^2),
\end{aligned} \tag{C8}$$

in accord with Eq.(2.61) of Ref.[21] for the balance of angular momentum.

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